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Matrix Form of the Iterated Binomial Transforms

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In a previous paper, we have estudied the Binomial transforms of the k–Fibonacci sequences and later we have studied the iterated Binomial transforms of these sequences. Now we study the matrix form of these four different binomial transforms of integer sequences in such a way that we can find a relation between the terms of the transformed sequence and the terms of the initial sequence. In this form we do not need to find the initial terms of the transformed sequence to find the following term from the application of the corresponding recurrence equation.
Later we will apply the obtained results to the particular case of the k–Fibonacci sequence.

Keywords: Binomial Transforms; Recurrence equation; Pascal matrix; k–Fibonacci numbers.

MSC 2000: 15A36, 11C20, 11B39.

1 Introduction

In the mathematical literature there are different generalizations of the classical Fibonacci sequence [1, 2]. In this paper we will use the definition of the k–Fibonacci sequence showed in [3, 4]: for

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a given integer number k, we define the k–Fibonacci sequence $F_k = \{F_{k,n}\}_{n\in\mathbf{N}}$ by mean of the recurrence relation

$$F_{k,n+1} = k\,F_{k,n} + F_{k,n-1} \text{ for } n \geq 1 \tag{1.1}$$

with the initial conditions $F_{k,0} = 0,\ F_{k,1} = 1$.

According to this definition, the general expression of the first terms of the k–Fibonacci sequence is $F_k = \{0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, \ldots\}$. In the particular case of $k = 1$ the classical Fibonacci sequence $F_1 = F = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$ is obtained while for $k = 2$ we get the known Pell sequence $F_2 = P = \{0, 1, 2, 5, 12, 29, \ldots\}$.

The characteristic equation corresponding to the previous definition (1.1) is $r^2 - k\,r - 1 = 0$ which characteristic positive root is $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$ and is known as Metallic Ratio [5]. Consequently, the well known Binnet Identity has the form $F_{k,n} = \dfrac{\sigma_k^n - (-\sigma_k)^{-n}}{\sigma_k + \sigma_k^{-1}}$.

A combinatorial formula to calculate the general term of the k–Fibonacci sequence is [3, 4, 6]: $F_{k,n} = \displaystyle\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-i}{i} k^{n-1-2i}$.

Finally, the generating function of the k–Fibonacci numbers is

$f_k(x) = \dfrac{x}{1 - kx - x^2}$.

On the other hand, the Pascal matrix is the infinite lower triangular matrix containing the binomial coefficients as its entries $P = \left(\binom{i-1}{j-1}\right)$ for $i, j = 1, 2\ldots$, and where i indicates the row and j is the column, and taking into account if $r > m$, then $\binom{m}{r} = 0$. Consequently, the Pascal matrix has the form

$$P = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

At last, the binomial transforms have been defined and studied in [7, 8, 9, 10, 11], whereas in [6, 12] they have been applied to the k–Fibonacci sequences getting the transformed sequences whose general terms are generated in the form that we indicate in the sequel:

- Binomial Transform: the terms of the binomial transformed sequence $B_k = \{b_{k,n}\}$ are defined by the combinatorial formula $b_{k,n} = \displaystyle\sum_{i=0}^{n} \binom{n}{i} F_{k,i}$ and verify the recurrence relation $b_{k,n+1} = (2+k)b_{k,n} - k\,b_{k,n-1}$ with $b_{k,0} = 0$ and $b_{k,1} = 1$.

  The recurrence relation of this relation is $r^2 = (2+k)r - k \to r^2 - (2+k)r + k = 0 \to r = \dfrac{(2+k) \pm \sqrt{k^2+4k+4-4k}}{2} = \dfrac{k \pm \sqrt{k^2+4}}{2} + 1$. If s_b is the positive characteristic root and $\sigma = \frac{k+\sqrt{k^2+4}}{2}$ is the positive characteristic root of the recurrence relation of the definition (1.1), then $s_b = \sigma + 1$.

In [6] is proven the generating function of the transformed sequence is $b(k,x) = \frac{x}{1-(k+2)x+k\,x^2}$

- k–Binomial Transform: the terms of the k–binomial transformed sequence $W_k = \{w_{k,n}\}$ are defined by the combinatorial formula $w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^n F_{k,i}$ and verify the recurrence relation $w_{k,n+1} = (2+k)kw_{k,n} - k^3 w_{k,n-1}$ with $w_{k,0} = 0$ and $w_{k,1} = k$.
If s_w is the positive characteristic root of the recurrence relation $r^2 = (2+k)r - k^3$, then $s_w = (\sigma+1)k$, where $\sigma = \frac{k+\sqrt{k^2+4}}{2}$.
Finally, the generating function of the k–binomial transformed sequence is [6] $w(k,x) = \frac{k\ x}{1-(k^2+2k)x+k^3x^2}$

- Rising k–Binomial Transform: the terms of the rising k–binomial transformed sequence $R_k = \{r_{k,n}\}$ are defined by the combinatorial formula $r_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^i F_{k,i}$ and verify the recurrence relation $r_{k,n+1} = (k^2+2)r_{k,n} - r_{k,n-1}$ with $r_{k,0} = 0$ and $r_{k,1} = k$. If s_r is the positive characteristic root of the recurrence relation $r^2 = (k^2+2)r - 1$, then $s_r = k\sigma + 1$ and its generating function is [6], $r(k,x) = \frac{k\ x}{1-(k^2+2)x+x^2}$

- Falling k–Binomial Transform: the terms of the falling k– binomial transformed sequence $F_k = \{f_{k,n}\}$ are defined by the combinatorial formula $f_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} F_{k,i}$ and verify the recurrence relation $f_{k,n+1} = 3kf_{k,n} - (2k^2-1)f_{k,n-1}$ with $f_{k,0} = 0$ and $f_{k,1} = 1$. If s_f is the positive characteristic root of the recurrence relation $r^2 = 3kr - (2k^2-1)$, then $s_f = \sigma + k$ being its generating function [6] $f(k,x) = \frac{x}{1-3kx+(2k^2-1)x^2}$

For $k = 1$, these four transforms are applied on the classical Fibonacci sequence and generate the same sequence $\{0, 1, 3, 8, 21, \ldots\}$, listed as A001906 in [13], OEIS from this point on.

We now turn to consider the matrix form of the previous four binomial transforms applied to a general integer sequence and then distinguish the results to the case of the k–Fibonacci sequence.

2 Matrix Form of the Binomial Transform

Let the integer sequence $A = \{a_0, a_1, a_2 \ldots\}$ be. If we apply the definition of Binomial Transform to this sequence, obtain the transformed sequence

$B = \{b_n\}$ where $b_n = \sum_{0}^{n} \binom{n}{i} a_i$.

In the particular case of the k–Fibonacci sequence, the sequences $B_k = \{b_{k,n}\} = \sum_{0}^{n} \binom{n}{i} F_{k,i}$ are obtained and from these, the unique indexed in OEIS are:

$B_1 = \{0, 1, 3, 8, 21, \ldots\} : A001906$, $B_2 = \{0, 1, 4, 14, 48, \ldots\} : A007070 \cup \{0\}$
$B_3 = \{0, 1, 5, 22, 95, \ldots\} : A116415 \cup \{0\}$, $B_4 = \{0, 1, 6, 32, 168, \ldots\} : A084326$

Binomial Transform of the integer sequence A can be expressed in matrix form as $B = P \cdot A$, being B the transformed binomial matrix $B = \{b_0, b_1, b_2 \ldots\}^T$ whereas A is the column matrix of the sequence, $A = (a_0, a_1, a_2 \ldots)^T$. Consequently

$$B = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}$$

This formula shows that the Binomial Transform is well defined by the Pascal matrix P.

But the Binomial Transform one can be applied to the obtained sequence again and we obtain the sequence $B^{(2)}$ that, evidently will be determined by the matrix P^2. Consequently it is $B^{(2)} = P^2 \cdot A$, and if we apply iteratly this process, will obtain the successive transformed binomial sequences of the integer sequence A which matrix equation is $B^{(r)} = P^r \cdot A$.

P^r is the r–th power of the Pascal matrix P and their entries $p_{i,j}^{(r)}$ verify the relation $p_{i,j}^{(r)} = \binom{i-1}{j-1} r^{i-j}$ [14, 15].

So, any element of the sequence obtained by mean of the r-th Binomial Transform of the integer sequence A can be expressed in function of the elements of this same sequence by mean of the formula

$$b_i^{(r)} = \sum_{j=1}^{i} \binom{i-1}{j-1} r^{i-j} a_{j-1}, \ r \geq 1 \tag{2.1}$$

In this form, it is not necessary to find the succesive transformed binomial sequences to find the transformed sequence of order r of an integer sequence but every element can be found directly using this formula (2.1).

If A is the k–Fibonacci sequence, the binomial transformed sequence of order r would take the form

$$B_k^{(r)} = \{b_i^{(r)}\} = \left\{ \sum_{j=1}^{i} \binom{i-1}{j-1} r^{i-j} F_{k,j-1} \right\}, \ r \geq 1$$

For instance, if we want to find the 5–th Binomial Transform of the 3–Fibonacci sequence $F_3 = \{0, 1, 3, 10, 33, 109, \ldots\}$ we find firstly P^5 using the formula (2.1) and obtain (for $n = 6$)

$$P_6^5 = \begin{pmatrix} 1 & & & & & \\ 5 & 1 & & & & \\ 25 & 10 & 1 & & & \\ 125 & 75 & 15 & 1 & & \\ 625 & 500 & 150 & 20 & 1 & \\ 3125 & 3125 & 1250 & 250 & 25 & 1 \end{pmatrix}.$$

Multiplying this matrix by the sequence F_3 we obtain its 5–th transformed binomial which first terms are $B_3^{(5)} = \{0, 1, 13, 130, 1183, 10309 \ldots\}$. This sequence is not indexed in OEIS.

In [12] one proves that the terms of the sequence $B_k^{(r)}$ verify the recurrence relation $b_{k,n+1}^{(r)} = (2r + k)b_{k,n}^{(r)} - (r^2 + rk - 1)b_{k,n-1}^{(r)}$ with $b_{k,0} = 0$, $b_{k,1} = 1$.

We must say that in this paper, and given the complexity of the used matrices, we have used the program MATHEMATICA to find this matrix as others that come later.

3 Matrix Form of the k–Binomial Transform

Let K be the diagonal matrix formed by the succesive powers of k, that is, $K = \text{D}iagonalMatrix\{k^0, k^1, k^2, \ldots, k^{n-1}\}$. Then, the k–Binomial Transform of the integer sequence A is the sequence $W_k = K \cdot P \cdot A$.

By the same reason that in the preceding section, as the k–Binomial Transform is fixed by the matrix $K \cdot P$, the sequence obtained by mean of the iterated application of this transform is fixed by the formula $W_k^{(r)} = (K \cdot P)^r \cdot A$. So, the r-th application of the k–Binomial Transform is fixed by the matrix $W_k^r = (K \cdot P)^r$ which form we have found using the program MATHEMATICA and it is

$$W_k^r = \left(\binom{i-1}{j-1} k^{r(j-1)+i-j} \left(\frac{k^r - 1}{k-1} \right)^{i-j} \right)$$

It is easy to apply this formula to the k–Fibonacci sequences, in such a way that if, for instance, we want to find the 6 first terms of the 4-Binomial Transform of order 3, we multiply the matrix

$$W_4^3 = \begin{pmatrix} 1 & & & & & \\ 84 & 64 & & & & \\ 7056 & 10752 & 4096 & & & \\ 592704 & 1354752 & 1032192 & 262144 & & \\ 49787136 & 151732224 & 173408256 & 88080384 & 16777216 & \\ 4182119424 & 15931883520 & 24277155840 & 18496880640 & 7046430720 & 1073741824 \end{pmatrix}$$

by the 4-Fibonacci sequence $F_4 = \{0, 1, 4, 17, 72, 305 \ldots\}$ and we will obtain the sequence $W_4^{(3)} = \{0, 64, 27136, 9939968, 3550691328, 1262321745920 \ldots\}$, that is, $W_4^{(3)} = 4^3\{0, 1, 424, 155312, 55479552, 19723777280 \ldots\}$.

The terms of the sequence $W^{(r)} = \{w_{k,n}^{(r)}\}$ verify the recurrence relation [12]

$$w_{k,n+1}^{(r)} = \left(k^{r+1} + 2\frac{k^r - 1}{k-1} k \right) w_{k,n}^{(r)} - \left(\frac{(k^r - 1)(k^{r+1} - 1)}{(k-1)^2} k^2 - k^{2r} \right) w_{k,n-1}^{(r)}$$

with $w_{k,0} = 0$, $w_{k,1} = k^r$.

4 Matrix Form of the k–Rising Binomial Transform

As in both previous sections, because the k–Rising Bionomial Transform is fixed by the matrix $P \cdot K$, the sequence obtained by mean of the iterated application of this transform verifies that $R_k^{(r)} = (P \cdot K)^r \cdot A$. So, the r-th application of the k–Rising Binomial Transform is fixed by the matrix $(P \cdot K)^r$ which form is

$$(P \cdot K)^r = \left(\binom{i-1}{j-1} k^{r(j-1)} \left(\frac{k^r - 1}{k-1} \right)^{i-j} \right)$$

For instance, the 5-th 2-Rising Binomial Transform of the 2-Fibonacci Sequence (Pell sequence), is the product of the matrix (for $n = 6$)

$$(P \cdot K)^5 = \begin{pmatrix} 1 & & & & & \\ 31 & 32 & & & & \\ 961 & 1984 & 1024 & & & \\ 29791 & 92256 & 95232 & 32768 & & \\ 923521 & 3813248 & 5904384 & 4063232 & 1048576 & \\ 28629151 & 147763360 & 305059840 & 314900480 & 162529280 & 33554432 \end{pmatrix}$$

by the sequence $F_2 = \{0, 1, 2, 5, 12, 29 \ldots\}$ and we obtain the sequence

$R_2^{(5)} = \{0, 32, 4032, 446560, 48521088, 5255815328 \ldots\}$, that is,
$R_2^{(5)} = 2^5\{0, 1, 126, 13955, 1516284, 164244229 \ldots\}$.

The terms of the sequence $R^{(r)} = \{r_{k,n}^{(r)}\}$ verify the recurrence relation [12]

$$r_{k,n+1}^{(r)} = \left(k^{r+1} + 2\frac{k^r - 1}{k-1}\right) r_{k,n}^{(r)} - \left(\frac{(k^r-1)(k^{r+1}-1)}{(k-1)^2} - k^r\right) r_{k,n-1}^{(r)}$$

with $r_{k,0} = 0$, $r_{k,1} = k^r$.

5 Matrix Form of the k–Falling Binomial Transform

The k–Falling Binomial Transform is fixed by the matrix $F = K{\cdot}P{\cdot}K^{-1}$ and it is possible to find that applying r times the obtained sequence, this transform is determined by the matrix that corresponds to its n-th power and that has the form $F^r = (K \cdot P \cdot K^{-1})^r = \left(\binom{i-1}{j-1} k^{r(j-1)} \left(\frac{k^r-1}{k-1}\right)^{i-j}\right)$.

For instance, the 3–th 5–Falling Binomial Transform is fixed by the matrix

$$F_5^3 = \begin{pmatrix} 1 & & & & & \\ 15 & 1 & & & & \\ 225 & 30 & 1 & & & \\ 3375 & 675 & 45 & 1 & & \\ 50625 & 13500 & 1350 & 60 & 1 & \\ 759375 & 253125 & 33750 & 2250 & 75 & 1 \end{pmatrix}$$

and by multiplying it by the sequence $F_5 = \{0, 1, 5, 26, 135, 701 \ldots\}$ causes the sequence $F_5^{(3)} = \{0, 1, 35, 926, 21945, 491201 \ldots\}$

The terms of the sequence $F_k^{(r)} = \{f_{k,n}^{(r)}\}$ verify the recurrence relation [12]

$$f_{k,n+1}^{(r)} = (2r+1)kf_{k,n}^{(r)} - ((r^2+r)k^2 - 1)f_{k,n-1}^{(r)} \quad \text{with } f_{k,0} = 0,\ f_{k,1} = 1$$

We will conclude this paper indicating that, as it is obvious, if $k = 1$, the four binomial transforms are the same in all cases: $B_1^{(r)} = W_1^{(r)} = R_1^{(r)} = F_1^{(r)}$ for $r \geq 1$.

6 Conclusions

We have applied the four different Binomial Transforms to the k-Fibonacci numbers of iterated form and have obtained the transformed sequences. Only a few of these transformed sequences are indexed in OEIS. We have found the Binnet Identity for all these sequences as well the generating function and the binomial formula for the general term.

Competing Interests

Author has declared that no competing interests exist.

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