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Coupled Fixed Point Results in Partially Ordered Fuzzy Metric Spaces

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Abstract

In the present paper we establish a coupled fixed point result in partially ordered fuzzy metric spaces by utilizing the control function. We obtain our results for *Hadžić* type t-norm. We also establish two lemmas and deduce a corollary. By an application of the fixed point theorem in fuzzy metric spaces, a corresponding result is obtained in metric spaces. Our work extends some existing results [1, 2, 3].

Keywords: Fixed point; fuzzy; metric space.

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1 Introduction and Mathematical Preliminaries

Fuzzy metric spaces had been introduced by Kramosil and Michalek in 1975 [4] which was later modified by George and Veeramani in their paper [5] for the purpose of introducing Hausdroff topology in the fuzzy metric space. There are also other definitions of fuzzy metric spaces, as,

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for examples, in the work of Kaleva and Seikkala [6], where fuzzy number has been introduced for this purpose. Fuzzy fixed point theory has developed in a very large way in this space introduced George and Veeramani. The successful development of metric fixed point theory in this space is largely due to the Hausdroff topology in this space. Some potential examples of fixed point theory in fuzzy metric spaces are in [6, 7, 8, 9, 10, 11, 12, 13, 14]. Particularly fuzzy extension of Banach contraction mapping principle has been done in works like [15, 16]. Wardowski [17] proved a fixed point theorem under certain conditions for a new fuzzy contraction obtained with help of a control function.

Coupled fixed point theorem, although introduced by Guo et al. [18], as early as in 1987 as attracted large attention metric fixed point theory after 2006 when a coupled contraction mapping theorem was established by Bhaskar et al. in [1]. Some references of this development are in [3, 19, 20, 21]. The first corrected extension of coupled fixed point result to fuzzy metric spaces was done by Zhu et al. [22]. This work was followed by several other works of the same topic by [7, 9, 23, 24].

The purpose here is to establish a coupled fixed point result in fuzzy metric spaces by utilizing the control function introduced by Wardowski. We obtain our results for *Hadžić* type t-norm, that is, t-norms which iterates are equi-continuous at 1.

Definition 1.1. [25, 26] A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a continuous t-norm if the following properties are satisfied:

i) $*$ is associative and commutative,

ii) $a * 1 = a$ for all $a \in [0,1]$,

iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$

iv) $*$ is continuous.

Some examples of continuous t-norm are $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$ and $a *_4 b = \max\{a + b - 1, 0\}$.

George and Veeramani in their paper [5] introduced the following definition of fuzzy metric space. We will be concerned only with this definition of fuzzy metric space.

Definition 1.2. [5] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

i) $M(x, y, t) > 0$,

ii) $M(x, y, t) = 1$ if and only if $x = y$,

iii) $M(x, y, t) = M(y, x, t)$,

iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and

v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology and is called the topology on X induced by the fuzzy metric M. This topology is metrizable as we indicated above.

Example 1.1. *[5] Let X be the set of all real numbers and d be the Euclidean metric by any set X and any metric d on X. Let $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. For each $t > 0$, $x, y \in X$, let*

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space.

Definition 1.3. [5] Let $(X, M, *)$ be a fuzzy metric space.

i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

The following lemma was proved by Grabiec [15] for fuzzy metric spaces defined by Kramosil et al. The proof is also applicable to the fuzzy metric space given in definition 1.2.

Lemma 1.2. *[15] Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x, y \in X$.*

Lemma 1.3. *[27] M is a continuous function on $X^2 \times (0, \infty)$.*

Let $(X, \preceq)$ be a partially ordered set and F be a mapping from X to itself. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$ [1].

Definition 1.4. [1] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is non-decreasing in its first argument and is non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$, for fixed $y \in X$ and, for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for fixed $x \in X$.

Definition 1.5. [28] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and, for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

Definition 1.6. [1] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$F(x, y) = x$ and $F(y, x) = y$

Definition 1.7. [17] Denote by H a family of mappings $\eta : (0, 1] \to [0, \infty)$ satisfying the following two conditions:

(H1) η transforms (0, 1] onto $[0, \infty)$;
(H2) $\forall s, t \in (0, 1]$, $[s < t \Rightarrow \eta(s) > \eta(t)]$ (i.e. η is strictly decreasing).
Note that (H1) and (H2) imply $\eta(1) = 0$ and $\eta(\alpha_n) \to 0$ whenever $\alpha_n \to 1$ as $n \to \infty$.

For example of η-function consider $\eta(t) = \frac{1}{t} - 1$ in $t \in (0, 1]$.

2 Main Results

Lemma 2.1. *Let $(X, M, *)$ be a fuzzy metric space and let $\eta \in H$. The sequences $\{x_n\}$ and $\{y_n\}$ in X are convergent to the points $x \in X$ and $y \in X$ if $\lim_{n\to\infty} \eta(M(x_n, x, t) * M(y_n, y, t)) = 0$ for all $t > 0$.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X . $\{x_n\}$ and $\{y_n\}$ in X are convergent to $x, y \in X$ respectively. Then using definition 1.4, $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} M(y_n, y, t) = 1$ for all $t > 0$. So, $\lim_{n\to\infty} (M(x_n, x, t) * M(y_n, y, t)) = 1$ for all $t > 0$. Now taking η on both sides, $\lim_{n\to\infty} \eta(M(x_n, x, t) * M(y_n, y, t)) = 0$ for all $t > 0$, by a property of η function.

Lemma 2.2. *The sequences $\{x_n\}$ and $\{y_n\}$ in X are Cauchy sequences if for each $0 < \epsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $\eta(M(x_n, x_m, t) * M(y_n, y_m, t)) \leq \epsilon$ for each $n, m \geq n_0$.*

Proof. If $\{x_n\}$ and $\{y_n\}$ in X are Cauchy sequences then for each $0 < \lambda < 1$ and $t > 0$ there exists positive integer n_0 such that $M(x_n, x_m, t) > 1 - \lambda, M(y_n, y_m, t) > 1 - \lambda$ for each $n, m \geq n_0$. So, $M(x_n, x_m, t) * M(y_n, y_m, t) > (1 - \lambda)$. then by $H2$, $\eta(M(x_n, x_m, t) * M(y_n, y_m, t)) < \eta(1 - \lambda)$. Put $\eta(1 - \lambda) = \epsilon$, so, $\eta(M(x_n, x_m, t) * M(y_n, y_m, t)) < \epsilon$ for each $n, m \geq n_0$..

Theorem 2.3. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $F : X \times X \to X$ be a mapping such that F has mixed monotone property and $\eta : (0, 1] \to [0, \infty)$ satisfies the properties $(H1)$ and $(H2)$. If the mapping F satisfies the condition:*

i) There exists x_0 and y_0 in X such that $\prod_{i=1}^{k}\{M(x_0, F(x_0, y_0), t_i) * M(y_0, F(y_0, x_0), t_i)\} \neq 0$, for all $k \in \mathbb{N}$, and $t_i \downarrow 0$,

ii) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, for all $r, s \in \{M(x_0, F(x_0, y_0), t) * M(y_0, F(y_0, x_0), t)$ for all $x_0, y_0 \in X, t > 0\}$,

iii) $\{\eta(M(x_0, F(x_0, y_0), t_i) * M(y_0, F(y_0, x_0), t_i)) : i \in \mathbb{N}\}$ is bounded for all x_0 and y_0 in X and any sequence $(t_i)_i \subset (0, \infty)$, $t_i \downarrow 0$,

iv) Finally,

$$\eta(M(F(x, y), F(u, v), t) * M(F(y, x), F(v, u), t)) \leq k \cdot \eta(M(x, u, t) * M(y, v, t)), \tag{2.1}$$

for all $x, y, u, v \in X$, $t > 0$ with $x \preceq u$ and $y \succeq v$ where $0 < k < 1$. Also suppose either

a) F is continuous or

b) X has the following properties:

i) if a non-decreasing sequence $\{x_n\} \to x$ then

$$x_n \preceq x \quad \text{for all} \quad n \geq 0, \tag{2.2}$$

ii) if a non-increasing sequence $\{y_n\} \to y$, then

$$y_n \succeq y \quad \text{for all} \quad n \geq 0. \tag{2.3}$$

If there are $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in X.

Proof. Starting with x_0, y_0 in X, we define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$x_1 = F(x_0, y_0) \quad \text{and} \quad y_1 = F(y_0, x_0),$$

$$x_2 = F(x_1, y_1) \quad \text{and} \quad y_2 = F(y_1, x_1),$$

and in general, for all $n \geq 0$,

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n). \tag{2.4}$$

Next, we prove that for all $n \geq 0$,

$$x_n \preceq x_{n+1} \tag{2.5}$$

and

$$y_n \succeq y_{n+1}. \tag{2.6}$$

From the conditions on x_0, y_0, we have $x_0 \preceq F(x_0, y_0) = x_1$ and $y_0 \succeq F(y_0, x_0) = y_1$. Therefore (2.5) and (2.6) hold for $n = 0$.

Let (2.5) and (2.6) hold for some $n = m$. As F has the mixed monotone property and $x_m \preceq x_{m+1}$, $y_m \succeq y_{m+1}$, it follows that

$$x_{m+1} = F(x_m, y_m) \preceq F(x_{m+1}, y_m) \quad \text{and} \quad F(y_{m+1}, x_m) \preceq F(y_m, x_m) = y_{m+1}. \tag{2.7}$$

Also, for the same reason, we have

$$F(x_{m+1}, y_m) \preceq F(x_{m+1}, y_{m+1}) = x_{m+2} \quad \text{and} \quad y_{m+2} = F(y_{m+1}, x_{m+1}) \preceq F(y_{m+1}, x_m). \tag{2.8}$$

So , from (2.7) and (2.8)

$$x_{m+1} \preceq x_{m+2}$$

and

$$y_{m+1} \succeq y_{m+2}$$

Then, by induction, (2.5) and (2.6) hold for all $n \geq 0$. Due to (2.1), from (2.4), for all $t > 0$, $n \geq 1$, we have

$$\begin{aligned} \eta(M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t)) &= \eta(M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t) * M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t)) \\ &\leq k\eta(M(x_{n-1}, x_n, t) * M(y_{n-1}, y_n, t)) \end{aligned}$$

that is,

$$\eta(M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t)) \leq k.\eta(M(x_{n-1}, x_n, t) * M(y_{n-1}, y_n, t)). \tag{2.9}$$

From (2.9), we can get for all $n \geq 1$, $t > 0$,

$$\begin{aligned} \eta(M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t)) &\leq k\eta(M(x_{n-1}, x_n, t) * M(y_{n-1}, y_n, t)) \\ &\leq k^2\eta(M(x_{n-2}, x_{n-1}, t) * M(y_{n-2}, y_{n-1}, t)) \\ &\leq k^3\eta(M(x_{n-3}, x_{n-2}, t) * M(y_{n-3}, y_{n-2}, t)) \\ &\vdots \\ &\leq k^n\eta(M(x_0, x_1, t) * M(y_0, y_1, t)). \end{aligned} \tag{2.10}$$

From the above, we have

$$M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t) \geq M(x_0, x_1, t) * M(y_0, y_1, t). \tag{2.11}$$

Now letting any $m, n \in \mathbb{N}$, $n > m$, $t > 0$, and let $(a_i)_i \in \mathbb{N}$ be a strictly decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$. From (2.11) and condition (i) of the theorem we have

$$\begin{aligned} M(x_m, x_n, t) * M(y_m, y_n, t) &\geq \{M(x_m, x_m, t - \sum_{i=m}^{n-1} a_i t) * M(x_m, x_n, \sum_{i=m}^{n-1} a_i t)\} \\ &\quad * \{M(y_m, y_m, \sum_{i=m}^{n-1} a_i t) * M(y_m, y_n, \sum_{i=m}^{n-1} a_i t)\} \\ &= \{1 * M(x_m, x_n, \sum_{i=m}^{n-1} a_i t)\} * \{1 * M(y_m, y_n, \sum_{i=m}^{n-1} a_i t)\} \\ &= M(x_m, x_n, \sum_{i=m}^{n-1} a_i t) * M(y_m, y_n, \sum_{i=m}^{n-1} a_i t) \\ &\geq \prod_{i=m}^{n-1} \{M(x_i, x_{i+1}, a_i t) * M(y_i, y_{i+1}, a_i t)\} \\ &\geq \prod_{i=m}^{n-1} \{M(x_0, x_1, a_i t) * M(y_0, y_1, a_i t)\}. \end{aligned} \tag{2.12}$$

By (2.12) and the condition (ii) of the theorem, we have

$$\begin{aligned} \eta(M(x_m, x_n, t) * M(y_m, y_n, t)) &\leq \eta(\prod_{i=m}^{n-1} \{M(x_i, x_{i+1}, a_i t) * M(y_i, y_{i+1}, a_i t)\}) \\ &\leq \sum_{i=m}^{n-1} \eta(M(x_i, x_{i+1}, a_i t) * M(y_i, y_{i+1}, a_i t))). \end{aligned} \tag{2.13}$$

From (2.10) and (2.13), we have

$$\eta(M(x_m, x_n, t) * M(y_m, y_n, t)) \leq \sum_{i=m}^{n-1} k^i \eta(M(x_0, x_1, a_i t) * M(y_0, y_1, a_i t)). \tag{2.14}$$

Here the sequence $\eta(M(x_0, x_1, a_i t) * M(y_0, y_1, a_i t))$ for all $i \in \mathbb{N}$, is increasing and by the condition (iii) of the theorem, we have a convergence of the series $\sum_{i=m}^{n-1} k^i \eta(M(x_0, x_1, a_i t) * M(y_0, y_1, a_i t))$. For given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=m}^{n-1} k^i \eta(M(x_0, x_1, a_i t) * M(y_0, y_1, a_i t)) < \epsilon \quad \text{for all} \quad m, n \geq n_0, \quad n > m. \tag{2.15}$$

From (2.14), we have

$$\eta(M(x_m, x_n, t) * M(y_m, y_n, t)) \leq \epsilon.$$

So by lemma (2.2), we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y. \tag{2.16}$$

Therefore,

$$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} F(x_n, y_n) = x$$

and

$$\lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} F(y_n, x_n) = y.$$

Also from (2.5), (2.6) and (2.16), we have that $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$ and $\{y_n\}$ is a non-increasing sequence with $y_n \to y$ as $n \to \infty$. Then, by (2.2) and (2.3), it follows that, for all $n \geq 0$,

$$x_n \preceq x \quad \text{and} \quad y_n \succeq y. \tag{2.17}$$

$$\begin{aligned}\eta(M(x_n, F(x,y), t) * M(y_n, F(y,x), t)) &= \eta(M(F(x_{n-1}, y_{n-1}), F(x,y), t) \\ &\quad *M(F(y_{n-1}, x_{n-1}), F(y,x), t)) \\ &\leq k\eta(M(x_n, x, t) * M(y_n, y, t)) \quad \text{(by (2.1) and (2.17))}\end{aligned}$$

Taking $n \to \infty$ on the both sides of the above inequality, we have

$$\lim_{n\to\infty} \eta(M(x_n, F(x,y), t) * M(y_n, F(y,x), t)) \leq 0,$$
$$x = \lim_{n\to\infty} x_n = F(x,y) \quad \text{and} \quad y = \lim_{n\to\infty} y_n = F(y,x),$$

that is, (x, y) is a coupled fixed point of F.
The following corollary is the coupled version of the Gragori and Sapena [16].

Corollary 2.4. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $F : X \times X \to X$ be a mapping such that F has mixed monotone property and $\eta : (0, 1] \to [0, \infty)$ satisfies the properties $(H1)$ and $(H2)$. If the mapping F satisfies the condition:*

i) There exists x_0 and y_0 in X such that

$$\prod_{i=1}^{k}\{M(x_0, F(x_0, y_0), t_i) * M(y_0, F(y_0, x_0), t_i)\} \neq 0$$

, for all $k \in \mathbb{N}$ and $t_i \downarrow 0$,

ii) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, for all $r, s \in \{M(x_0, F(x_0, y_0), t) * M(y_0, F(y_0, x_0), t)$ for all $x_0, y_0 \in X, t > 0\}$,

iii) $\{\eta(M(x_0, F(x_0, y_0), t_i) * M(y_0, F(y_0, x_0), t_i)) : i \in \mathbb{N}\}$ is bounded for all x_0 and y_0 in X and any sequence $(t_i)_i \subset (0, \infty)$, $t_i \downarrow 0$,

iv) $\left(\dfrac{1}{M(F(x,y), F(u,v), t) * M(F(y,x), F(v,u), t)} - 1\right) \leq k.\left(\dfrac{1}{M(x,u,t) * M(y,v,t)} - 1\right)$,

for all $x, y, u, v \in X$, $t > 0$ with $x \preceq u$ and $y \succeq v$ where $0 < k < 1$. Also suppose either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$

(ii) if a non-increasing sequence $\{y_n\} \to y$,then $y_n \succeq y$ for all $n \geq 0$.

If there are $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in X.

Proof. Putting $\eta(t) = \frac{1}{t} - 1$, then proof follows by Theorem 2.3.

3 Application in Metric Space

In this section we apply Theorem 2.1 of the previous section to obtain a coupled coincidence point result in partially ordered metric spaces. Several existing results [1, 2, 3] are hereby extended.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mappings such that F has the mixed monotone property and satisfies the following condition:*

$$\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\} \leq k\max\{d(x,u),d(y,v)\} \tag{3.1}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, where $0 < k < 1$. Suppose $F(X \times X) \subseteq X$. Also suppose either

(a) F is continuous or

(b) X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.

If there are $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in X.

Proof. For all $x, y \in X$ and $t > 0$, we define

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

and $a * b = \min\{a, b\}$. Then, as noted earlier, $(X, M, *)$ is a complete fuzzy metric space.
Let $\eta(t) = \frac{1}{t} - 1$ in $t \in (0, 1]$.
Next we show that the inequality (2.1) implies (3.1). From (2.1), for all $t > 0$, $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, we have

$$\Big(\frac{1}{\min\{\frac{t}{t+d(F(x,y),F(u,v))},\frac{t}{t+d(F(y,x),F(v,u))}\}}-1\Big) < k\Big(\frac{1}{\min\{\frac{t}{t+d(x,u)},\frac{t}{t+d(u,v)}\}}-1\Big),$$

that is,

$$\Big(\frac{1}{\frac{t}{t+\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}}} - 1\Big) < k\Big(\frac{1}{\frac{t}{t+\max\{d(x,u),d(y,v)\}}} - 1\Big),$$

that is,

$$\max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}, < k\max\{d(x,u),d(y,v)\}$$

which is (3.1). The proof is then completed as an application of Theorem 2.3.

4 Conclusion

In this paper we use a control function to obtain coupled fixed point theorems in Fuzzy metric spaces. There can also be other uses of the control function in similar problems. Further the arguments in this paper may be extended to prove n-tupled fixed point results.

Competing Interests

The authors declare that no competing interests exist.

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