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An Improved DGMRES Algorithm for Computing the Drazin-inverse Solution of Singular Linear Systems

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Abstract

Krylov subspace methods have been considered to solve singular linear systems $Ax = b$. One of these methods is the DGMRES method. DGMRES is an algorithm to solve the Drazin–inverse solution of the large scale and sparse consistent or inconsistent singular linear systems with with arbitrary index. In this paper, we present an improved version of this algorithm. Numerical experiments show that computation time is significantly less than that of computation time obtained by the DGMRES algorithm.

Keywords: Singular linear systems; DGMRES method; drazin-inverse solution; index; krylov subspace methods.

1 Introduction

Consider the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{N\times N}$ is a singular matrix and $\mathrm{ind}(A)$ is arbitrary. Here $\mathrm{ind}(A)$, the index of A is the smallest nonnegative integer a such that $\mathrm{rank}(A^{a+1}) = \mathrm{rank}(A^a)$. The Drazin–inverse (see [1] or [2]), denoted by A^D, of A is the unique matrix satisfying

$$AA^DA = A^D, \quad AA^D = A^DA, \quad A^{a+1}A^D = A^a,$$

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where a is the index of A. We recall that the Drazin-inverse solution of (1.1) is the vector $A^D b$, where A^D is the Drazin–inverse of the singular matrix A. Iterative methods for singular linear systems include stationery iterative methods, semi–iterative methods and Krylove subspase methods. Stationery iterative methods for singular linear systems have been well studied in [3], [4], [5], [6], [7], [8] and [9]. Sidi [10], Wei and Wu [11] firstly presented the Krylov subspace methods for singular linear systems with arbitrary index, works for consistent singular linear systems are perhaps newly emerged some modified GMRES methods in [12], [13] and [8]. Works for consistent or inconsistent singular linear systemss with an arbitrary index are presented in [14], [10], [15] and [16]. The Drazin-inverse has various applications in the theory of finite Markov chains [2], the study of singular differential and difference equations [2], the investigation of Cesaro-Neumanniterations (see [17], [18], [19], [20], [21], [22] and [23]), cryptography [24], iterative methods in numerical analysis [25], multibody system dynamics [26] and others. It is well known that the representations of the Drazin– inverse of matrices are very important not only in matrix theory, but also in singular differential and difference equations, probability statistical, numerical analysis, game theory, econometrics, control theory and so on (see [2] and [1]), and also singular systems with arbitrary index arise naturally in Markov chain modelling (see [27] and [28]).

The problem of finding the solution of the form $A^D b$ for (1.1) is very common in the literature and many different techniques have been developed in order to solve it. In [29], A. Sidi proposed a general approach to Krylov subspace methods for computing Drazin-inverse solution. And then, he gave several Krylov subspace methods of Arnoldi, DGCR and Lancoze types. Moreover in [14] and [10], Sidi has continued to drive two Krylov subspace methods for computing $A^D b$. One is DGMRES method, which is implementation of the DGCR method for singular systems which is analogues to GMRES for non-singular systems. Other is DBI-CG method which is Lanczos type algorithm. DGMRES, just like, GMRES method, is a stable numerically and economical computationally and storage wise method. DBI-CG method, also just like BI-CG for non–singular systems, is a fast algorithm, but when we need a high accuracy, the algorithm is invalid. In the present paper, we suggests the IMDGMRES algorithm which is another implementation of DGMRES, for solving the singular linear system (1.1) with arbitrary index. By numerical examples, we show that the computation time of IMDGMRES algorithm is substantially less than that of DGMRES algorithm.

The paper is organized as follows. In section 2, we will give a review of DGMRES. In section 3, we will derive the IMDGMRES algorithm. In Section 4 the results of some numerical examples are given. Section 5 is devoted to concluding remarks.

2 DGMRES Algorithm

DGMRES method is a Krylov subspace method for computing the Drazin–inverse solution of consistent or inconsistent linear systems (1.1) (see [29] and [10]). In this method, there are not any restriction on the matrix A. Thus, in general, A is non-hermitian, $a := \mathrm{ind}(A)$ is arbitrary, and the spectrum of A can have any shape. DGMRES starts with an initial vectors x_0 and generates a sequence of vectors $x_0, x_1, \ldots$, as

$$x_m = x_0 + \sum_{i=1}^{m-a} c_i A^{a+i-1} r_0, \qquad r_0 = b - Ax_0.$$

Then

$$r_m = b - Ax_m = b - \sum_{i=1}^{m-a} c_i A^{a+i} r_0.$$

The Krylov subspace we will use is

$$\mathcal{K}_{m-a}\{A; A^a r_0\} = \mathrm{span}\{A^a r_0, A^{a+1} r_0, \ldots, A^{m-1} r_0\}.$$

The vector x_m produced by DGMRES satisfies

$$\|A^a r_m\|_2 = \min_{x\in x_0+\mathcal{K}_{m-a}\{A;A^a r_0\}} \|A^a(b-Ax)\|_2. \tag{2.1}$$

As $x_m = x_0+\sum_{i=1}^{m-a} c_i A^{a+i-1} r_0$, we starts by orthogonalizing the krylov vectors $A^a r_0, A^{a+1} r_0, \ldots,$, by the Arnoldi–Gram–Schmidt process (see [30] and [31]), carried out numerically like the modified Gram–Schmidt process:

For $i = 1, 2, \ldots,$ do
Compute $h_{ji} = (v_j, Av_i), \quad j = 1, 2, \ldots, i.$
Compute $\hat{v}_i = Av_i - \sum_{j=1}^{i} v_j h_{ji}$.
Let $h_{i+1,i} = \|\hat{v}_i\|_2$ and set $v_{i+1} = \hat{v}_i/h_{i+1,i.}$

Consequently, we have a set of orthonormal vectors $v_1, v_2, \ldots,$ that satisfies

$$Av_i = \sum_{j=1}^{i+1} v_j h_{ji}, \quad i = 1, 2, \ldots, \tag{2.2}$$

as long as $i \le q-1$, where q is the degree of the minimal polynomial of A with respect to $A^a r_0$, hence with respect to v_1. Furthermore, for each k,

$$\begin{aligned} \operatorname{span}\{v_1, v_2, \ldots, v_k\} &= \operatorname{span}\{\mathrm{A}^a \mathrm{r}_0, \mathrm{A}^{a+1}\mathrm{r}_0, \ldots, \mathrm{A}^{k+a-1}\mathrm{r}_0\} \\ &= \mathcal{K}_k(A; A^a r_0). \end{aligned} \tag{2.3}$$

If we now define the $N \times k$ matrix $\hat{V}_k$ by

$$\hat{V}_k = [v_1|v_2|\ldots|v_k], \quad k = 1, 2, \ldots, \tag{2.4}$$

then, for $m \le m_0$ (for definition of m_0 see [29] and [10]), we can write

$$x_m = x_0 + \hat{V}_{m-a}\xi_m, \quad \text{for some } \xi_\mathrm{m} \in \mathbb{C}^{\mathrm{m-a}} \tag{2.5}$$

and we need to determine ξ_m. Since $r_m = r_0 + A\hat{V}_{m-a}\xi_m$, we have

$$A^a r_m = A^a r_0 + A^{a+1}\hat{V}_{m-a}\xi_m = \beta v_1 - A^{a+1}\hat{V}_{m-a}\xi_m. \tag{2.6}$$

Next, provided $k \le q-1$, from (2.2) we can write

$$A\hat{V}_k = \hat{V}_{k+1}\bar{H}_k; \qquad \bar{H}_k = \begin{bmatrix} h_{11} & h_{12} & \cdots & \cdots & h_{1k} \\ h_{21} & h_{22} & \cdots & \cdots & h_{2k} \\ 0 & h_{32} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & h_{kk} \\ 0 & \cdots & \cdots & 0 & h_{k+1,k} \end{bmatrix}. \tag{2.7}$$

Note that $\bar{H}_k \in \mathbb{C}^{(k+1)\times k}$ and $\bar{H}_k$ has full rank when $k \le q-1$ [10]. Now, by using (2.6), (2.7), and $\hat{V}_{m+1}^*\hat{V}_{m+1} = I_{(m+1)\times(m+1)}$ we can reduce the $n \times (m-a)$ least squares problem of (2.1) to the $(m+1)\times(m-a)$ least squares problem

$$\|A^a r_m\|_2 = \|\hat{V}_{m+1}(\beta e_1 - \hat{H}_m\xi_m)\|_2 = \min_{\xi\in\mathbb{C}^{m-a}} \|\hat{V}_{m+1}(\beta e_1 - \hat{H}_m\xi)\|_2 = \min_{\xi\in\mathbb{C}^{m-a}} \|\beta e_1 - \hat{H}_m\xi\|_2, \tag{2.8}$$

where

$$\hat{H}_m = \bar{H}_m\bar{H}_{m-1}\ldots\bar{H}_{m-a}, \tag{2.9}$$

and $\hat{H}_m \in \mathbb{C}^{(m+1)\times(m-a)}$. Note that n is normally very large and $m \ll n$, which implies that the problem in (2.8) is very small. The minimization problem (2.8) is accomplished by using the QR decomposition of $\hat{H}_m$. For more details we refer the reader to [29] and [10]. We now summarize the steps of DGMRES for the solution of a linear system $Ax = b$, where A is singular and $a = \text{ind(A)}$ is known.

Algorithm 1 DGMRES algorithm

1. Pick x_0 and compute $r_0 = b - Ax_0$ and $A^a r_0$.
2. Compute $\beta = \|A^a r_0\|$ and set $v_1 = \beta^{-1}(A^a r_0)$.
3. Orthogonalize the Krylov vectors $A^a r_0, A^{a+1} r_0, \ldots,$ via the Arnoldi-Gram-Schmidt process
   carried out like the modified Gram-Schmidt process:
   For $i = 1, 2, \ldots,$ do
   Compute $h_{ji} = (v_j, Av_i), j = 1, 2, \ldots, i$.
   Compute $\hat{v}_i = Av_i - \sum_{j=1}^{i} v_j h_{ji}$.
   Let $h_{i+1,i} = \|\hat{v}_i\|$ and set $v_{i+1} = \hat{v}_i / h_{i+1,i}$.
   (The vectors $v_1, v_2, \ldots,$ obtained by this way form an orthonormal set.)
4. For $k = 1, 2, \ldots,$ form the matrices $\hat{V}_k \in \mathbb{C}^{N\times k}$ and $\bar{H}_k \in \mathbb{C}^{(k+1)\times k}$
   as defined in (2.4) and (2.7), respectively.
5. For $m = a + 1, ...,$ form the matrix $\hat{H}_m = \bar{H}_m \bar{H}_{m-1} \ldots \bar{H}_{m-a}$.
6. Compute the QR factorization of $\hat{H}_m : \hat{H}_m = Q_m R_m; Q_m \in \mathbb{C}^{(m+1)\times(m-a)}$ and $R_m \in \mathbb{C}^{(m-a)\times(m-a)}$.
   (R_m is upper triangular.)
7. Solve the (upper triangular) system $R_m \xi_m = \beta(Q_m^* e_1)$, where $e_1 = [1, 0, \ldots, 0]$.
8. Compute $x_m = x_0 + \hat{V}_{m-a}\xi_m$ (then $\|A^a r_m\| = \beta\sqrt{1 - \|Q_m^* e_1\|^2}$).

3 IMDGMRES Method

In this section, we suggest a simple modification of the DGMRES algorithm for solving the Drazin–inverse solution of linear systems (1.1). In this method, there are not any restriction on the matrix A. Thus, in general, A is non-hermitian, $a := \text{ind}(A)$ is arbitrary, and the spectrum of A can have any shape.

From (2.8) we have

$$\|A^a r_m\|_2 = \min_{\xi\in\mathbb{R}^{m-a}} \|\beta e_1 - \hat{H}_m \xi\|_2, \tag{3.1}$$

where $\hat{H}_m = \bar{H}_m \bar{H}_{m-1} \ldots \bar{H}_{m-a}$. For solving the least squares problems, suppose that the matrix $Q_m \in \mathbb{R}^{(m+1)\times(m+1)}$ is a multiplication of the Givens to annihilate the strictly lower part of $\bar{H}_m \in \mathbb{R}^{(m+1)\times m}$ and $\bar{g}_{m+1} = \beta e_1 = [\beta \quad 0 \quad \ldots \quad 0]^T$. From (2.1) we can obtains

$$\begin{aligned}
\|A^a r_m\|_2^2 &= \|\bar{g}_{m+1} - \bar{H}_m \bar{H}_{m-1} \ldots \bar{H}_{m-a}\xi_{m-a}\|_2^2 \\
&= \|Q_m(\bar{g}_{m+1}) - Q_m(\bar{H}_m)\bar{H}_{m-1} \ldots \bar{H}_{m-a}\xi_{m-a}\|_2^2 \\
&= \left\| \begin{pmatrix} g_m \\ \gamma_{m+1} \end{pmatrix} - \begin{pmatrix} R_m \\ 0 \end{pmatrix} \bar{H}_{m-1} \ldots \bar{H}_{m-a}\xi_{m-a} \right\|_2^2 \\
&= \|g_m - R_m \bar{H}_{m-1}\bar{H}_{m-2} \ldots \bar{H}_{m-a}\xi_{m-a}\|_2^2 + |\ \gamma_{m+1}\ |^2 .
\end{aligned}$$

Since $R_m \in \mathbb{R}^{m\times m}$ is an upper triangular matrix and $\bar{H}_{m-1} \in \mathbb{R}^{m\times m-1}$ is upper Hessenberg matrix, consequently $R_m \bar{H}_{m-1} \in \mathbb{R}^{m\times m-1}$ is a upper Hessenberg matrix. Suppose that $H_{m-1}^* = R_m \bar{H}_{m-1}$.

Also, let $Q_{m-1} \in \mathbb{R}^{m\times m}$ which is obtained by multiplying Givens rotations. As a result $Q_{m-1}H^*$ is an upper triangular matrix. So we have

$$\begin{aligned}\|A^a r_m\|_2^2 &= \|g_m - H^*_{m-1}\bar{H}_{m-2}\ldots\bar{H}_{m-a}\xi_{m-a}\|_2^2 + \mid \gamma_{m+1}\mid^2\\ &= \|Q_{m-1}g_m - (Q_{m-1}H^*_{m-1})\bar{H}_{m-2}\ldots\bar{H}_{m-a}\xi_{m-a}\|_2^2 + \mid \gamma_{m+1}\mid^2 .\\ &= \left\| \begin{pmatrix} g_{m-1}\\ \gamma_m\end{pmatrix} - \begin{pmatrix} R_{m-1}\\ 0\end{pmatrix}\bar{H}_{m-2}\ldots\bar{H}_{m-a}\xi_{m-a}\right\|_2^2 + \mid \gamma_{m+1}\mid^2\\ &= \|g_{m-1} - H^*_{m-2}\bar{H}_{m-3}\ldots\bar{H}_{m-a}\xi_{m-a}\|_2^2 + \mid \gamma_m\mid^2 + \mid \gamma_{m+1}\mid^2 .\end{aligned}$$

After $a+1$ repeating this process we have

$$\|A^a r_m\|_2^2 = \|g_{m-a} - R_{m-a}\xi_{m-a}\|_2^2 + \mid \gamma_{m-a+1}\mid^2 + \ldots + \mid \gamma_m\mid^2 + \mid \gamma_{m+1}\mid^2, \qquad (3.2)$$

where $g_{m-a} = [\gamma_1 \quad \gamma_2 \ldots \gamma_{m-a}]^T$, $R_{m-a} \in \mathbb{R}^{(m-a)\times(m-a)}$ and also $\xi_{m-a} \in \mathbb{R}^{m-a}$. To minimize (3.2) is sufficient we choose $\xi_{m-a} = R^{-1}_{m-a}g_{m-a}$.

The advantage of this approach is that we do not need to build the matrix $\hat{H}_m$ and also this method reduces computation time by approximately two times which is a significant advantage of the method IMDGMRES compared to the method DGMRES. Now, we summarize the steps of the new method, called IMDGMRES method, for comuting the Drazin–inverse solution of singular linear systems $Ax = b$ as follows.

Algorithm 2 IMDGMRES algorithm

1. Pick x_0 and compute $r_0 = b - Ax_0$ and $A^a r_0$.
2. Compute $\beta = \|A^a r_0\|$ and set $v_1 = \beta^{-1}(A^a r_0)$.
3. Orthogonalize the Krylov vectors $A^a r_0, A^{a+1}r_0, \ldots,$ via the Arnoldi-Gram-Schmidt process
 carried out like the modified Gram-Schmidt process:
 For $i = 1, 2, \ldots,$ do
 Compute $h_{ji} = (v_j, Av_i), j = 1, 2, \ldots, i.$
 Compute $\hat{v}_i = Av_i - \sum_{j=1}^{i} v_j h_{ji}$.
 Let $h_{i+1,i} = \|\hat{v}_i\|$ and set $v_{i+1} = \hat{v}_i/h_{i+1,i}$.
 (The vectors $v_1, v_2, \ldots,$ obtained by this way form an orthonormal set.)
4. For $k = 1, 2, \ldots,$ form the matrices $\hat{V}_k \in \mathbb{C}^{N\times k}$ and $\bar{H}_k \in \mathbb{C}^{(k+1)\times k}$as defined in (2.4)
 and (2.7), respectively.
5. Set $H^*_m = \overline{H}_m$
6. For $k = m, m-1, \cdots, m-a$ Do :
7. Compute the QR factorization of H^*_k : $H^*_k = Q_k\overline{R}_k$,
 where $Q_k \in \mathbb{R}^{(k+1)\times(k+1)}$ and $\overline{R}_k \in \mathbb{R}^{(k+1)\times k}$
8. Set $H^*_{k-1} = \overline{R}_k\overline{H}_{k-1}$, $\overline{g}_k = \begin{cases} Q_k(\beta e_1), & If\ k = m,\\ Q_k\overline{g}_{k+1}, & Otherwise,\end{cases}$
9. EndDo
10. Compute ξ_{m-a}, the minimizer of $\|g_{m-a} - R_{m-a}\xi\|_2$ and $x_m = x_0 + \widehat{V}_{m-a}\xi_{m-a}$.

4 Numerical Examples

To compare the behavior of the proposed IMDGMRES method discussed in the previous section with the DGMRES method, we present in this section numerical results for two examples. Our examples, which have a singular coefficient matrix, are derived by the finite difference method for elliptic partial differential equations. The numerical computations are performed in MATLAB (R2010b) with double precision. The results were obtains by running the code on an Intel Core 2 (Duo) 8400 Processor running 2.26GHz with 3 GB of RAM memory using Windows Vista professional 32-bit operating system. The initial vector x_0 is the zero vector. All the tests were stopped as soon as

$$Re = \frac{\|x_n - A^D b\|_\infty}{\|A^D b\|_\infty} \leq 10^{-8}.$$

Example 4.1. We will compute the linear system $Ax = b$ by discretizing Poisson equation with Neumann boundary conditions:

$$\begin{cases} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u(x,y) = f(x,y), & (x,y) \in \Omega = [0,1] \times [0,1] \\ \frac{\partial}{\partial n}u(x,y) = \varphi(x,y) & x, y \in \partial\Omega. \end{cases}$$

This linear system has also been computed by Sidi [10] for testing DGMRES algorithm. The problem has also been considered by Hank and Hochbruck [32] for testing the Chebyshev-type semi-iterative method.

Let M be an odd integer, we discretize the Poisson equation on a uniform grid of mesh size $h = 1/M$ via central differences, and then by taking the unknowns in the red-black order we obtain the system $Ax = b$, where the $(M+1)^2 \times (M+1)^2$ nonsymmetric matrix A is as follows

$$A = \left[\begin{array}{cccccccccccccccc}
4I & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & T_1 & -2I & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 4I & \ddots & & & & & \vdots & -I & T_2 & -I & 0 & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & & & \vdots & 0 & -I & T_1 & -I & 0 & & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & & & \vdots & \vdots & 0 & -I & T_2 & -I & 0 & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & \ddots & \ddots & \ddots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & & \ddots & 4I & 0 & \vdots & & & & 0 & -I & T_1 & -I \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 4I & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -2I & T_2 \\
T_2 & -2I & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 4I & o & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-I & T_1 & -I & 0 & & & & \vdots & 0 & 4I & \ddots & & & & & \vdots \\
0 & -I & T_2 & -I & 0 & & & \vdots & \vdots & \ddots & \ddots & \ddots & & & & \vdots \\
\vdots & 0 & -I & T_1 & -I & 0 & & \vdots & \vdots & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & 0 & -I & T_2 & -I & \vdots & & & & & \ddots & 4I & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -2I & T_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 4I
\end{array}\right] \tag{4.1}$$

Here, I and 0 denote, respectively, the $(M+1)/2 \times (M+1)/2$ identity and zero matrices and the $(M+1)/2 \times (M+1)/2$ matrices T_1 and T_2 are given by

$$T_1 = \begin{bmatrix} -2 & 0 & \cdots & \cdots & 0 \\ -1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & & -1 & 0 \\ 0 & \cdots & 0 & -1 & -1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} -1 & -1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & -1 & -1 \\ 0 & \cdots & \cdots & 0 & -2 \end{bmatrix}.$$

The numerical experiment was done for $M = 31, 63, 127$. **Example 4.2.** As shown in[32], applying 5−point central differences to the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + d\frac{\partial U}{\partial x} = f(x, y), \quad 0 < x, y < 1,$$

over the unit square $\Omega = (0, 1) \times (0, 1)$ with the periodic boundary condition:

$$u(x, 0) = u(x, 1), \quad u(0, y) = u(1, y)$$

yields a singular system with a nonsymmetric coefficient matrix. The mesh size is chosen as $h = 1/m$ for Ω, so that the resulting system has the following $n \times n$ coefficient matrix (where $n = m^2$):

$$A := \frac{1}{h^2} \begin{bmatrix} D_m & I_m & \cdots & & & I_m \\ I_m & D_m & I_m & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & I_m & D_m & I_m \\ I_m & & & & I_m & D_m \end{bmatrix}, \tag{4.2}$$

Here I_m is the $m \times m$ unit matrix and D_m the $m \times m$ matrix given by

$$D_m := \begin{bmatrix} -4 & \alpha_+ & & & \alpha_- \\ \alpha_- & -4 & \alpha_+ & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_- & -4 & \alpha_+ \\ \alpha_+ & & & \alpha_- & -4 \end{bmatrix},$$

where $\alpha_\pm = 1 \pm \frac{dh}{2}$. The numerical experiment was done for $d = 0.1$, $d = 0.3$, $d = 0.5$, and $m = 60$.

For the matrix A of both (4.1) and (4.2) the identity $Ae = A^T e = 0$ holds, so that $\text{Null}(A) = \text{Null}(A^T) = \text{Span}\{e\}$, where $e = (1, 1, \ldots, 1)^T$. Furthermore, $\text{ind}(A) = 1$, as mentioned in [32, 10]. Even if the continuous problem has a solution, the discretized problem need not be consistent. Here, we consider only the Drazin-inverse solution of the system for arbitrary right side b, not necessarily related to f and φ.

As [16], we first construct a consistent system with known solution $\hat{s} \in R(A)$ via $\hat{s} = Ay$, where $y = [0, \ldots, 0, 1]^T$. Then we perturb $A\hat{s}$, the right-hand side of $Ax = A\hat{s} = \hat{b}$, with a constant multiple of the null space vector e and we obtain the right-hand side

$$b = \hat{b} + \delta\frac{e}{\|e\|_2}.$$

Consequently the system $Ax = \hat{b} + \delta\frac{e}{\|e\|_2}$ is solved for x. The perturbation parameter δ is selected as 10^{-2} in our experiments.

For these examples, the solution we are looking for is the vector $\hat{s}$, whose components are zeros except

$$\hat{s}_{2\hat{M}^2-\hat{M}} = -1, \quad \hat{s}_{2\hat{M}^2-1} = -1, \quad \hat{s}_{2\hat{M}^2} = -2, \quad \hat{s}_{4\hat{M}^2} = 4, \quad \text{where } \hat{M} = (M+1)/2$$

for Example 4.1 and except

$$\hat{s}_m = 1, \quad \hat{s}_{m^2-m} = 1, \quad \hat{s}_{m^2-m+1} = \alpha_-, \quad \hat{s}_{m^2-1} = \alpha_+, \quad \hat{s}_{m^2} = 4$$

for Example 4.2.

In Tables 1-4, we give the number of iterations (Its), the CPU time (Time) required for convergence, and the error (Error) for the DGMRES and IMDGMRES methods. As shown in Tables 1-4 the IMDGMRES algorithm is effective and less expensive than the DGMRES algorithm.

Table 1. Application of IMDGMRES implementation to the consistent singular system for Example 4.1

Size of A	1024×1024			4096×4096			16384×16384		
Method	Its	Time	Error	Its	Time	Error	Its	Time	Error
DGMRES	127	0.58	$9.66e-009$	127	2.10	$9.84e-009$	126	6.82	$9.87e-009$
IMDGMRES	127	0.43	$9.66e-009$	310	1.28	$9.84e-009$	126	3.67	$9.87e-009$

Table 2. Application of IMDGMRES implementation to the inconsistent singular system for Example 4.1

Size of A	1024×1024			4096×4096			16384×16384		
Method	Its	Time	Error	Its	Time	Error	Its	Time	Error
DGMRES	127	0.62	$9.66e-009$	127	2.02	$9.84e-009$	126	7.06	$9.87e-009$
IMDGMRES	127	0.48	$9.66e-009$	127	1.16	$9.84e-009$	126	3.79	$9.87e-009$

Table 3. Application of IMDGMRES implementation to the consistent singular system for Example 4.2 with $m = 60$(n=3600)

A	0.1			0.3			0.5		
Method	Its	Time	Error	Its	Time	Error	Its	Time	Error
DGMRES	128	2.09	$9.61e-009$	128	1.73	$9.83e-009$	129	2.41	$9.68e-009$
IMDGMRES	128	1.16	$9.61e-009$	128	1.13	$9.83e-009$	129	1.19	$9.68e-009$

Table 4. Application of IMDGMRES implementation to the inconsistent singular systemfor Example 4.2 with $m = 60$(n=3600)

A	0.1			0.3			0.5		
Method	Its	Time	Error	Its	Time	Error	Its	Time	Error
DGMRES	128	2.09	$9.61e-009$	128	2.06	$9.83e-009$	129	2.41	$9.68e-009$
IMDGMRES	128	1.61	$9.61e-009$	128	1.26	$9.83e-009$	129	1.19	$9.68e-009$

5 Conclusion

In this paper, we have presented an improved versition of the DGMRES algorithm, called IMDGMRES, for computing the Drazin-inverse solution of singular linear equations with arbitrary index. Numerical experiments show that the Drazin-inverse solution obtained by this method is its computation time is less than that of solution obtained by the DGMRES method. So, we can conclude that the IMDGMRES algorithm is a robust and efficient tool for computing the Drazin-inverse solution of singular linear equations with arbitrary index.

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Competing Interests

The author declares that no competing interests exist.

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