Asian Journal of Mathematics and Computer Research

Volume 31, Issue 3, Page 1-11, 2024; Article no.AJOMCOR.12235
ISSN: 2395-4205 (P), ISSN: 2395-4213 (O)

Some Non-Linear Periodic Systems of Difference Equations

Kemal Uslu [a++*] and Halil Şeran [a]

[a] *Department of Mathematics, Faculty of Science, Selcuk University, Konya-42000, Turkey.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: https://doi.org/10.56557/ajomcor/2024/v31i38777

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://prh.ikprress.org/review-history/12235

Original Research Article

Received: 10/05/2024
Accepted: 15/07/2024
Published: 18/07/2024

Abstract

In this study, we considered two different difference equation systems. We showed that one of these systems has 6 periods and the other has 8 periods. Then, we obtained the equilibrium points of these systems and examined some behaviors of the system depending on the equilibrium points.

Keywords: Difference equations; systems of difference equations; nonlinear periodic systems of difference equations.

1.Introduction

A system of difference equations often tells us about a problem in daily life, science or engineering [1-4], [5-8]. In this respect, the equilibrium points of the difference equation system and how it will behave at these

[++] *Professor;*
**Corresponding author: Email: kuslu@selcuk.edu.tr;*

Cite as: *Uslu, Kemal, and Halil Şeran. 2024. "Some Non-Linear Periodic Systems of Difference Equations". Asian Journal of Mathematics and Computer Research 31 (3):1-11. https://doi.org/10.56557/ajomcor/2024/v31i38777.*

equilibrium points are extremely important [9-10]. The purpose of this study: Using many of the models we have mentioned above, our primary goals are to first create a different difference equation system and investigate the balance points and periodicity of this system, as well as examine the behavior of the system at the balance points of the system. In line with this goal, by using [4], this study will focus on two different models given below:

$$
\begin{aligned}
x_{n+1} &= \frac{y_{n-1}}{y_n(x_{n-2}+y_{n-2}+z_{n-2})} + \frac{1}{(x_{n-1}+y_{n-1}+z_{n-1})}, \\
y_{n+1} &= \frac{1}{(x_{n-1}+y_{n-1}+z_{n-1})}, \\
z_{n+1} &= \frac{1}{x_{n-1}-y_{n-1}} - \frac{y_{n-1}}{y_n(x_{n-2}+y_{n-2}+z_{n-2})} - \frac{2}{(x_{n-1}+y_{n-1}+z_{n-1})},\ (n \geq 0)
\end{aligned}
\tag{1.1}
$$

with initial values x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0 ($x_{-1}-y_{-1}\neq 0$, $x_0-y_0\neq 0$, $x_{-2}+y_{-2}+z_{-2}\neq 0$, $x_{-1}+y_{-1}+z_{-1}\neq 0$, $x_0+y_0+z_0\neq 0$) $\in \mathbb{R}$-{0} and

$$
\begin{aligned}
x_{n+1} &= \frac{y_{n-1}}{y_n(x_{n-3}+y_{n-3}+z_{n-3})} + \frac{1}{(x_{n-2}+y_{n-2}+z_{n-2})}, \\
y_{n+1} &= \frac{1}{(x_{n-2}+y_{n-2}+z_{n-2})}, \\
z_{n+1} &= \frac{1}{x_{n-2}-y_{n-2}} - \frac{y_{n-1}}{y_n(x_{n-3}+y_{n-3}+z_{n-3})} - \frac{2}{(x_{n-2}+y_{n-2}+z_{n-2})},\ (n \geq 0)
\end{aligned}
\tag{1.2}
$$

with initial values x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0, z_{-3}, z_{-2}, z_{-1}, z_0 ($x_{-1}-y_{-1}\neq 0$, $x_0-y_0\neq 0$, $x_{-3}+y_{-3}+z_{-3}\neq 0$, $x_{-2}+y_{-2}+z_{-2}\neq 0$, $x_{-1}+y_{-1}+z_{-1}\neq 0$, $x_0+y_0+z_0\neq 0$) $\in \mathbb{R}$-{0}.

Firstly, we give basic preliminary definitions and a theorem. Let I_1, I_2 and I_3 be some intervals of real numbers and let $F_1: I_1\times I_2\times I_3 \to I_1$, $F_2: I_1\times I_2\times I_3 \to I_2$ and $F_3: I_1\times I_2\times I_3 \to I_3$ be three continuously differentiable functions. For every initial condition $(x_s, y_s, z_s)\in I_1\times I_2\times I_3$, it is obvious that the system of difference equations (1.3)

$$
\begin{aligned}
x_{n+1} &= F_1(x_n, y_n, z_n) \\
y_{n+1} &= F_2(x_n, y_n, z_n) \\
z_{n+1} &= F_3(x_n, y_n, z_n)
\end{aligned}
\tag{1.3}
$$

has a unique solution $\{x_n, y_n, z_n\}$.

Now, we can give some definitions and theorem in literatüre:

Definition 1.1. A solution $\{x_n, y_n, z_n\}$ of the system of difference equations (1.3) is periodic if there exist a positive integer *p* such that $x_{n+p}=x_n$, $y_{n+p}=y_n$, $z_{n+p}=z_n$ the smallest such positive integer *p* is called the prime period of the solution of difference equation system (1.3).

Definition 1.2. A point $(x, y, z)\in I_1\times I_2\times I_3$ is called an equilibrium point of system (1.3), if

$x = F_1(x, y, z), \quad y = F_2(x, y, z), \quad z = F_3(x, y, z)$ [1,2].

Theorem 1.1. Let $J(x, y, z)$ be Jacobian matrix of system of difference equations (1.3) at the equilibrium point (x, y, z) and $P(\lambda)$ denote the characteristics polynomial of matrix $J(x, y, z)$. Then the followings are true:

a) If all roots of $P(\lambda)$ lie inside the open unit disk $|\lambda|<1$, then the equilibrium point (x, y, z) is asymptotically stable.
b) If all roots of $P(\lambda)$ have absolute value greater than one, then the equilibrium point (x, y, z) is repeller [1,2].

2.Main Results

In this section all results have been obtained by using [3,4]. The following theorems show us the period of solutions of the systems (1.1) and (1.2).

Theorem 2.1. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.1) with initial values $x_{-2} = p$, $x_{-1} = q$, $x_0 = r$, $y_{-2} = s$, $y_{-1} = t$, $y_0 = u$, $z_{-2} = k$, $z_{-1} = l$, $z_0 = m$ ($x_{-1} - y_{-1} \neq 0$, $x_0 - y_0 \neq 0$, $x_{-2} + y_{-2} + z_{-2} \neq 0$, $x_{-1} + y_{-1} + z_{-1} \neq 0$, $x_0 + y_0 + z_0 \neq 0$) $\in \mathbb{R}$-{0}. Then all solutions of the system (1.1) are periodic with period 6.

Proof: From the system (1.1), it is obtained the following equalities by iteration method:

$$x_{n+1} = \frac{y_{n-1}}{y_n(x_{n-2} + y_{n-2} + z_{n-2})} + \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})},$$
$$y_{n+1} = \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})},$$
$$z_{n+1} = \frac{1}{x_{n-1} - y_{n-1}} - \frac{y_{n-1}}{y_n(x_{n-2} + y_{n-2} + z_{n-2})} - \frac{2}{(x_{n-1} + y_{n-1} + z_{n-1})},$$

$$x_{n+2} = y_n + \frac{1}{(x_n + y_n + z_n)}, \quad y_{n+2} = \frac{1}{(x_n + y_n + z_n)}, \quad z_{n+2} = \frac{1}{x_n - y_n} - y_n - \frac{2}{(x_n + y_n + z_n)},$$
$$x_{n+3} = \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})} + x_{n-1} - y_{n-1}, \quad y_{n+3} = x_{n-1} - y_{n-1},$$
$$z_{n+3} = \frac{y_n(x_{n-2} + y_{n-2} + z_{n-2})}{y_{n-1}} - \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})} - 2(x_{n-1} - y_{n-1})$$

$$x_{n+4} = \frac{1}{(x_n + y_n + z_n)} + x_n - y_n, \quad y_{n+4} = x_n - y_n, \quad z_{n+4} = \frac{1}{y_n} - \frac{1}{(x_n + y_n + z_n)} - 2(x_n - y_n)$$

$$x_{n+5} = (x_{n-1} - y_{n-1}) + \frac{y_{n-1}}{y_n(x_{n-2} + y_{n-2} + z_{n-2})}, \quad y_{n+5} = \frac{y_{n-1}}{y_n(x_{n-2} + y_{n-2} + z_{n-2})},$$
$$z_{n+5} = (x_{n-1} + y_{n-1} + z_{n-1}) - (x_{n-1} - y_{n-1}) - \frac{2y_{n-1}}{y_n(x_{n-2} + y_{n-2} + z_{n-2})}$$

$x_{n+6} = x_n, \quad y_{n+6} = y_n, \quad z_{n+6} = z_n$

Thus all solutions of the system (1.1) are periodic with 6 period.

Theorem 2.2. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.2) with initial values $x_{-3} = a$, $x_{-2} = b$, $x_{-1} = c$, $x_0 = d$, $y_{-3} = p$, $y_{-2} = q$, $y_{-1} = r$, $y_0 = s$, $z_{-3} = t$, $z_{-2} = u$, $z_{-1} = v$, $z_0 = w$ ($x_{-2} - y_{-2} \neq 0$, $x_{-1} - y_{-1} \neq 0$, $x_0 - y_0 \neq 0$, $x_{-3} + y_{-3} + z_{-3} \neq 0$, $x_{-2} + y_{-2} + z_{-2} \neq 0$, $x_{-1} + y_{-1} + z_{-1} \neq 0$, $x_0 + y_0 + z_0 \neq 0$) $\in \mathbb{R}$-{0}. Then all solutions of the system (1.2) are periodic with period 8.

Proof: From the system (1.2), it is obtained the following equalities by iteration method:

$$x_{n+1} = \frac{y_{n-1}}{y_n(x_{n-3} + y_{n-3} + z_{n-3})} + \frac{1}{(x_{n-2} + y_{n-2} + z_{n-2})},$$
$$y_{n+1} = \frac{1}{(x_{n-2} + y_{n-2} + z_{n-2})},$$
$$z_{n+1} = \frac{1}{x_{n-2} - y_{n-2}} - \frac{y_{n-1}}{y_n(x_{n-3} + y_{n-3} + z_{n-3})} - \frac{2}{(x_{n-2} + y_{n-2} + z_{n-2})},$$

$$x_{n+2} = y_n + \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})}, \quad y_{n+2} = \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})}, \quad z_{n+2} = \frac{1}{x_{n-1} - y_{n-1}} - y_n - \frac{2}{(x_{n-1} + y_{n-1} + z_{n-1})},$$
$$x_{n+3} = \frac{1}{(x_{n-2} + y_{n-2} + z_{n-2})} + \frac{1}{(x_n + y_n + z_n)}, \quad y_{n+3} = \frac{1}{(x_n + y_n + z_n)},$$
$$z_{n+3} = \frac{1}{x_n - y_n} - \frac{1}{(x_{n-2} + y_{n-2} + z_{n-2})} - \frac{2}{(x_n + y_n + z_n)},$$

$$x_{n+4} = \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})} + (x_{n-2} - y_{n-2}), \quad y_{n+4} = (x_{n-2} - y_{n-2}),$$
$$z_{n+4} = \frac{y_n(x_{n-3} + y_{n-3} + z_{n-3})}{y_{n-1}} - \frac{1}{(x_{n-1} + y_{n-1} + z_{n-1})} - 2(x_{n-2} - y_{n-2})$$

$$x_{n+5} = \frac{1}{(x_n + y_n + z_n)} + (x_{n-1} - y_{n-1}), \quad y_{n+5} = (x_{n-1} - y_{n-1}), \quad z_{n+5} = \frac{1}{y_n} - \frac{1}{(x_n + y_n + z_n)} - 2(x_{n-1} - y_{n-1})$$

$$x_{n+6} = (x_{n-2} - y_{n-2}) + (x_n - y_n), \quad y_{n+6} = (x_n - y_n),$$
$$z_{n+6} = (2y_{n-2} + z_{n-2}) - 2(x_n - y_n)$$

$$x_{n+7} = (x_{n-1} - y_{n-1}) + \frac{y_{n-1}}{y_n(x_{n-3} + y_{n-3} + z_{n-3})}, \quad y_{n+7} = \frac{y_{n-1}}{y_n(x_{n-3} + y_{n-3} + z_{n-3})},$$
$$z_{n+7} = (2y_{n-1} + z_{n-1}) - \frac{2y_{n-1}}{y_n(x_{n-3} + y_{n-3} + z_{n-3})},$$

$$x_{n+8} = x_n, \quad y_{n+8} = y_n, \quad z_{n+8} = z_n$$

Thus all solutions of the system (1.2) are periodic with 8 period.

Theorem 2.3. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.1) with initial values $x_{-2} = p$, $x_{-1} = q$, $x_0 = r$, $y_{-2} = s$, $y_{-1} = t$, $y_0 = u$, $z_{-2} = k$, $z_{-1} = l$, $z_0 = m$ ($x_{-1} - y_{-1} \neq 0$, $x_0 - y_0 \neq 0$, $x_{-2} + y_{-2} + z_{-2} \neq 0$, $x_{-1} + y_{-1} + z_{-1} \neq 0$, $x_0 + y_0 + z_0 \neq 0$) $\in \mathbb{R}$-{0}. In this case, for $n \geq 0$, all solutions of (1.1) are

$$x_{6n+1} = \frac{t}{u(p+s+k)} + \frac{1}{(q+t+l)}, \quad y_{6n+1} = \frac{1}{(q+t+l)}, \quad z_{6n+1} = \frac{1}{q-t} - \frac{t}{u(p+s+k)} - \frac{2}{(q+t+l)},$$

$$x_{6n+2} = u + \frac{1}{(r+u+m)}, \quad y_{6n+2} = \frac{1}{(r+u+m)}, \quad z_{6n+2} = \frac{1}{r-u} - u - \frac{2}{(r+u+m)},$$

$$x_{6n+3} = \frac{1}{(q+t+l)} + q - t, \quad y_{6n+3} = q - t, \quad z_{6n+3} = \frac{u(p+s+k)}{t} - \frac{1}{(q+t+l)} - 2(q-t)$$

$$x_{6n+4} = \frac{1}{(r+u+m)} + r - u, \quad y_{6n+4} = r - u, \quad z_{6n+4} = \frac{1}{u} - \frac{1}{(r+u+m)} - 2(r-u)$$

$$x_{6n+5} = (q-t) + \frac{t}{u(p+s+k)}, \quad y_{6n+5} = \frac{t}{u(p+s+k)}, \quad z_{6n+5} = (q+t+l) - (q-t) - \frac{2t}{u(p+s+k)}$$

$$x_{6n+6} = r, \quad y_{6n+6} = u, \quad z_{6n+6} = m.$$

Proof: Let us use the principle of mathematical induction on *n*. For *n*=0, it is easy to see. Assume that it is true for all positive integers *n*. From the system (1.1), it is obtained the following equalities:

$$x_{6n+7} = \frac{y_{6n+5}}{y_{6n+6}(x_{6n+4} + y_{6n+4} + z_{6n+4})} + \frac{1}{(x_{6n+5} + y_{6n+5} + z_{6n+5})} = \frac{t}{u(p+s+k)} + \frac{1}{(q+t+l)},$$

$$y_{6n+7} = \frac{1}{(x_{6n+4} + y_{6n+4} + z_{6n+4})} = \frac{1}{(q+t+l)},$$

$$z_{6n+7} = \frac{1}{(x_{6n+5} - y_{6n+5})} - \frac{y_{6n+5}}{y_{6n+6}(x_{6n+4} + y_{6n+4} + z_{6n+4})} - \frac{2}{(x_{6n+5} + y_{6n+5} + z_{6n+5})}$$

$$z_{6n+7} = \frac{1}{q-t} - \frac{t}{u(p+s+k)} - \frac{2}{(q+t+l)},$$

$$x_{6n+8} = y_{6n+6} + \frac{1}{x_{6n+6} + y_{6n+6} + z_{6n+6}} = u + \frac{1}{(r+u+m)} =, \quad y_{6n+8} = \frac{1}{x_{6n+6} + y_{6n+6} + z_{6n+6}} = \frac{1}{(r+u+m)},$$

$$z_{6n+8} = \frac{1}{x_{6n+6} - y_{6n+6}} - y_{6n+6} - \frac{2}{x_{6n+6} + y_{6n+6} + z_{6n+6}} = \frac{1}{r-u} - u - \frac{2}{(r+u+m)},$$

$$x_{6n+9}=\frac{1}{x_{6n+5}+y_{6n+5}+z_{6n+5}}+x_{6n+5}-y_{6n+5}=\frac{1}{(q+t+l)}+q-t,\quad y_{6n+9}=x_{6n+5}-y_{6n+5}=q-t,$$

$$z_{6n+9}=\frac{y_{6n+6}(x_{6n+4}+y_{6n+4}+z_{6n+4})}{y_{6n+5}}-\frac{1}{x_{6n+5}+y_{6n+5}+z_{6n+5}}-2(x_{6n+5}-y_{6n+5})$$

$$z_{6n+9}=\frac{u(p+s+k)}{t}-\frac{1}{(q+t+l)}-2(q-t)$$

$$x_{6n+10}=\frac{1}{x_{6n+6}+y_{6n+6}+z_{6n+6}}+x_{6n+6}-y_{6n+6}=\frac{1}{(r+u+m)}+r-u,\quad y_{6n+10}=x_{6n+6}-y_{6n+6}=r-u,$$

$$z_{6n+10}=\frac{1}{y_{6n+6}}-\frac{1}{x_{6n+6}+y_{6n+6}+z_{6n+6}}-2(x_{6n+6}-y_{6n+6})=\frac{1}{u}-\frac{1}{(r+u+m)}-2(r-u)$$

$$x_{6n+11}=(x_{6n+5}-y_{6n+5})+\frac{y_{6n+5}}{y_{6n+6}(x_{6n+4}+y_{6n+4}+z_{6n+4})}=(q-t)+\frac{t}{u(p+s+k)},$$

$$y_{6n+11}=\frac{y_{6n+5}}{y_{6n+6}(x_{6n+4}+y_{6n+4}+z_{6n+4})}=\frac{t}{u(p+s+k)},$$

$$z_{6n+11}=(x_{6n+5}+y_{6n+5}+z_{6n+5})-(x_{6n+5}-y_{6n+5})-\frac{2y_{6n+5}}{y_{6n+6}(x_{6n+4}+y_{6n+4}+z_{6n+4})}$$

$$z_{6n+11}=(q+t+l)-(q-t)-\frac{2t}{u(p+s+k)}$$

$$x_{6n+12}=x_{6n+6}=r,\quad y_{6n+12}=y_{6n+6}=u,\quad z_{6n+12}=z_{6n+6}=m.$$

Theorem 2.4. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.2) with initial values $x_{-3}=a$, $x_{-2}=b$, $x_{-1}=c$, $x_0=d$, $y_{-3}=p$, $y_{-2}=q$, $y_{-1}=r$, $y_0=s$, $z_{-3}=t$, $z_{-2}=u$, $z_{-1}=v$, $z_0=w$ ($x_{-2}-y_{-2}\neq 0$, $x_{-1}-y_{-1}\neq 0$, $x_0-y_0\neq 0$, $x_{-3}+y_{-3}+z_{-3}\neq 0$, $x_{-2}+y_{-2}+z_{-2}\neq 0$, $x_{-1}+y_{-1}+z_{-1}\neq 0$, $x_0+y_0+z_0\neq 0$) $\in \mathbb{R}$-{0}. In this case, for $n\geq 0$, all solutions of (1.2) are

$$x_{8n+1}=\frac{r}{s(a+p+t)}+\frac{1}{(b+q+u)},\quad y_{8n+1}=\frac{1}{(b+q+u)},\quad z_{8n+1}=\frac{1}{b-q}-\frac{r}{s(a+p+t)}-\frac{2}{(b+q+u)},$$

$$x_{8n+2}=s+\frac{1}{(c+r+v)},\quad y_{8n+2}=\frac{1}{(c+r+v)},\quad z_{8n+2}=\frac{1}{c-r}-s-\frac{2}{(c+r+v)},$$

$$x_{8n+3}=\frac{1}{(b+q+u)}+\frac{1}{(d+s+w)},\quad y_{8n+3}=\frac{1}{(d+s+w)},\quad z_{8n+3}=\frac{1}{d-s}-\frac{1}{(b+q+u)}-\frac{2}{(d+s+w)},$$

$$x_{8n+4}=\frac{1}{(c+r+v)}+(b-q),\quad y_{8n+4}=(b-q),\quad z_{8n+4}=\frac{s(a+p+t)}{r}-\frac{1}{(c+r+v)}-2(b-q)$$

$$x_{8n+5}=\frac{1}{(d+s+w)}+(c-r),\quad y_{8n+5}=(c-r),\quad z_{8n+5}=\frac{1}{s}-\frac{1}{(d+s+w)}-2(c-r)$$

$$x_{8n+6}=(b-q)+(d-s),\quad y_{8n+6}=(d-s),\quad z_{8n+6}=(2q+u)-2(d-s)$$
$$x_{8n+7}=(c-r)+\frac{r}{s(a+p+t)},\quad y_{8n+7}=\frac{r}{s(a+p+t)},\quad z_{8n+7}=(2r+v)-\frac{2r}{s(a+p+t)},$$
$$x_{8n+8}=d,\quad y_{8n+8}=s,\quad z_{8n+8}=w.$$

Proof: Let us use the principle of mathematical induction on *n*. For *n*=0, it is easy to see. Assume that it is true for all positive integers *n*. From the system (1.2), it is obtained the following equalities:

$$x_{8n+9}=\frac{y_{8n+7}}{y_{8n+8}(x_{8n+5}+y_{8n+5}+z_{8n+5})}+\frac{1}{(x_{8n+6}+y_{8n+6}+z_{8n+6})}=\frac{r}{s(a+p+t)}+\frac{1}{(b+q+u)},$$
$$y_{8n+9}=\frac{1}{(x_{8n+6}+y_{8n+6}+z_{8n+6})}=\frac{1}{(b+q+u)},$$
$$z_{8n+9}=\frac{1}{x_{8n+6}-y_{8n+6}}+\frac{y_{8n+7}}{y_{8n+8}(x_{8n+5}+y_{8n+5}+z_{8n+5})}-\frac{2}{(x_{8n+6}+y_{8n+6}+z_{8n+6})}$$
$$z_{8n+9}=\frac{1}{b-q}+\frac{r}{s(a+p+t)}-\frac{2}{(b+q+u)},$$

$$x_{8n+10}=y_{8n+8}+\frac{1}{(x_{8n+7}+y_{8n+7}+z_{8n+7})}=s+\frac{1}{(c+r+v)},$$
$$y_{8n+10}=\frac{1}{(x_{8n+7}+y_{8n+7}+z_{8n+7})}=\frac{1}{(c+r+v)},$$
$$z_{8n+10}=\frac{1}{x_{8n+7}-y_{8n+7}}-y_{8n+8}-\frac{2}{(x_{8n+7}+y_{8n+7}+z_{8n+7})}=\frac{1}{c-r}-s-\frac{2}{(c+r+v)},$$
$$x_{8n+11}=\frac{1}{(x_{8n+6}+y_{8n+6}+z_{8n+6})}+\frac{1}{(x_{8n+8}+y_{8n+8}+z_{8n+8})}=\frac{1}{(b+q+u)}+\frac{1}{(d+s+w)},$$
$$y_{8n+11}=\frac{1}{(x_{8n+8}+y_{8n+8}+z_{8n+8})}=\frac{1}{(d+s+w)},$$
$$z_{8n+11}=\frac{1}{x_{8n+8}-y_{8n+8}}-\frac{1}{(x_{8n+6}+y_{8n+6}+z_{8n+6})}-\frac{2}{(x_{8n+8}+y_{8n+8}+z_{8n+8})}$$
$$z_{8n+11}=\frac{1}{d-s}-\frac{1}{(b+q+u)}-\frac{2}{(d+s+w)},$$

$$x_{8n+12}=\frac{1}{(x_{8n+7}+y_{8n+7}+z_{8n+7})}+(x_{8n+6}-y_{8n+6})=\frac{1}{(c+r+v)}+(b-q),$$
$$y_{8n+12}=(x_{8n+6}-y_{8n+6})=(b-q),$$
$$z_{8n+12}=\frac{y_{8n+8}(x_{8n+5}+y_{8n+5}+z_{8n+5})}{y_{8n+7}}-\frac{1}{(x_{8n+7}+y_{8n+7}+z_{8n+7})}-2(x_{8n+6}-y_{8n+6})$$
$$z_{8n+12}=\frac{s(a+p+t)}{r}-\frac{1}{(c+r+v)}-2(b-q)$$

$$x_{8n+13}=\frac{1}{(x_{8n+8}+y_{8n+8}+z_{8n+8})}+(x_{8n+7}-y_{8n+7})=\frac{1}{(d+s+w)}+(c-r),$$
$$y_{8n+13}=(x_{8n+7}-y_{8n+7})=(c-r),$$
$$z_{8n+13}=\frac{1}{y_{8n+8}}-\frac{1}{(x_{8n+8}+y_{8n+8}+z_{8n+8})}-2(x_{8n+7}-y_{8n+7})=\frac{1}{s}-\frac{1}{(d+s+w)}-2(c-r)$$

$$x_{8n+14}=(x_{8n+6}-y_{8n+6})+(x_{8n+8}-y_{8n+8})=(b-q)+(d-s),$$
$$y_{8n+14}=(x_{8n+8}-y_{8n+8})=(d-s),$$
$$z_{8n+14}=(2y_{8n+6}+z_{8n+6})-2(x_{8n+8}-y_{8n+8})=(2q+u)-2(d-s)$$

$$x_{8n+15}=(x_{8n+7}-y_{8n+7})+\frac{y_{8n+7}}{y_{8n+8}(x_{8n+5}+y_{8n+5}+z_{8n+5})}=(c-r)+\frac{r}{s(a+p+t)},$$
$$y_{8n+15}=\frac{y_{8n+7}}{y_{8n+8}(x_{8n+5}+y_{8n+5}+z_{8n+5})}=\frac{r}{s(a+p+t)},$$
$$z_{8n+15}=(2y_{8n+7}+z_{8n+7})-\frac{2y_{8n+7}}{y_{8n+8}(x_{8n+5}+y_{8n+5}+z_{8n+5})}=(2r+v)-\frac{2r}{s(a+p+t)},$$
$$x_{8n+16}=x_{8n+8}=d,\quad y_{8n+16}=y_{8n+8}=s,\quad z_{8n+16}=z_{8n+8}=w$$

Theorem 2.5. The difference equation systems (1.1) and (1.2) have two equilibrium points which are $\left(A,\frac{A}{2},\frac{4-3A^2}{2A}\right)$, $\left(-A,\frac{-A}{2},\frac{3A^2-4}{2A}\right)\in I_1\times I_2\times I_3$, where I_1, I_2 and I_3 are some intervals of real numbers and $A\in IR-\{0\}$.

Proof: For the equilibrium points of the systems (1.1) and (1.2), we can write the following equalities

$$x=F_1(x,y,z)=\frac{y}{y(x+y+z)}+\frac{1}{(x+y+z)},\quad y=F_2(x,y,z)=\frac{1}{(x+y+z)},$$
$$z=F_3(x,y,z)=\frac{1}{x-y}-\frac{y}{y(x+y+z)}-\frac{2}{(x+y+z)}.$$

From above equations, we obtain the results

$$(x,y,z)=\left(A,\frac{A}{2},\frac{4-3A^2}{2A}\right),\quad (x,y,z)=\left(-A,\frac{-A}{2},\frac{3A^2-4}{2A}\right).$$

Theorem 2.6. The Jacobian matrix of the system (1.3) in the equilibrium points which are $\left(A,\frac{A}{2},\frac{4-3A^2}{2A}\right)$, $\left(-A,\frac{-A}{2},\frac{3A^2-4}{2A}\right)$ follows

$$J(x,y,z)=\begin{pmatrix} \frac{-A^2}{2} & \frac{-A^2}{2} & \frac{-A^2}{2} \\ \frac{-A^2}{4} & \frac{-A^2}{4} & \frac{-A^2}{4} \\ \frac{-4}{A^2}+\frac{3A^2}{4} & \frac{4}{A^2}+\frac{3A^2}{4} & \frac{3A^2}{4} \end{pmatrix}.$$

Proof: Jacobian matrix at any point (x,y,z) is

$$J(x,y,z)=\begin{pmatrix} \frac{\partial F_1(x,y,z)}{\partial x} & \frac{\partial F_1(x,y,z)}{\partial y} & \frac{\partial F_1(x,y,z)}{\partial z} \\ \frac{\partial F_2(x,y,z)}{\partial x} & \frac{\partial F_2(x,y,z)}{\partial y} & \frac{\partial F_2(x,y,z)}{\partial z} \\ \frac{\partial F_3(x,y,z)}{\partial x} & \frac{\partial F_3(x,y,z)}{\partial y} & \frac{\partial F_3(x,y,z)}{\partial z} \end{pmatrix}.$$

From the system (1.3), we have

$$J(x,y,z)=\begin{pmatrix} \frac{-2}{(x+y+z)^2} & \frac{-2}{(x+y+z)^2} & \frac{-2}{(x+y+z)^2} \\ \frac{-1}{(x+y+z)^2} & \frac{-1}{(x+y+z)^2} & \frac{-1}{(x+y+z)^2} \\ \frac{-1}{(x-y)^2}+\frac{3}{(x+y+z)^2} & \frac{1}{(x-y)^2}+\frac{3}{(x+y+z)^2} & \frac{3}{(x+y+z)^2} \end{pmatrix}.$$

For the equilibrium points which are $\left(A,\frac{A}{2},\frac{4-3A^2}{2A}\right)$, $\left(-A,\frac{-A}{2},\frac{3A^2-4}{2A}\right)$, Jacobian matrix is

$$J(x,y,z)=\begin{pmatrix} \frac{-A^2}{2} & \frac{-A^2}{2} & \frac{-A^2}{2} \\ \frac{-A^2}{4} & \frac{-A^2}{4} & \frac{-A^2}{4} \\ \frac{-4}{A^2}+\frac{3A^2}{4} & \frac{4}{A^2}+\frac{3A^2}{4} & \frac{3A^2}{4} \end{pmatrix}.$$

Let $P(\lambda)$ denote the characteristics polynomial of matrix $J(x,y,z)$. In this case, it is obvious

$$P(\lambda)=\left|J(x,y,z)-\lambda I\right|=\begin{vmatrix} \frac{-A^2}{2}-\lambda & \frac{-A^2}{2} & \frac{-A^2}{2} \\ \frac{-A^2}{4} & \frac{-A^2}{4}-\lambda & \frac{-A^2}{4} \\ \left(\frac{-4}{A^2}+\frac{3A^2}{4}\right) & \left(\frac{4}{A^2}+\frac{3A^2}{4}\right) & \frac{3A^2}{4}-\lambda \end{vmatrix}=-\lambda^3+\lambda\cdot$$

The roots of $P(\lambda)$: $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = -1$. Thus we can write following results:

a) All roots of $P(\lambda)$ don't lie inside the open disk $|\lambda| < 1$. As a result of this, the equilibrium points $(x, y, z) = \left(A, \frac{A}{2}, \frac{4-3A^2}{2A} \right)$ and $(x, y, z) = \left(-A, \frac{-A}{2}, \frac{3A^2-4}{2A} \right)$ are not asymptotically stable.

b) Because all roots of $P(\lambda)$ don't have absolute value greater than one, the equilibrium points $(x, y, z) = \left(A, \frac{A}{2}, \frac{4-3A^2}{2A} \right)$ and $(x, y, z) = \left(-A, \frac{-A}{2}, \frac{3A^2-4}{2A} \right)$ are not repeller .

3. Conclusions

Many different features related to the difference equation systems considered in this study that have not been examined in this study can be examined.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Acknowledgements

This study is related to Halil Seran's master thesis.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Clark D, Kulenovic MRS. A coupled system of rational difference equations, Computers and Mathematics with Applications. 2002;43:849-867.

[2] Nasri M, Dehghan M,Douraki MJ, Mathias R. Study of a system of non-linear difference equations arising in a deterministic model for HIV infection, Applied Mathematics and Computation. 2005;171:1306-1330.

[3] Uslu K, Taskara N, Hekimoglu O. On the periodicity and stability conditions of a non-linear system, The First International Conference on Mathematics and Statistics, American University of Sharjah, UAE. 2010;110:18-21.

[4] Kılıklı G. On the solutions of a system of the rational difference equations, The graduate school of natural and applied science od Selcuk University, Ms thesis; 2011.

[5] Chunhua F. Dynamic Behavior of a Delayed Competitive-Cooperative Model, Asian Journal of Mathematics and Computer Research. 2023;30(3):1-9.
Article no. AJOMCOR.11564,

[6] Ndung'u Reuben M. A Sensitivity Analysis of a Mathematical Model for the Transmission of Endemic Malaria under Periodic Climatic Conditions, Asian Journal of Mathematics and Computer Research. 2023;30(4):1-15.
Article no. AJOMCOR.11665,

[7] Zhou Y, Yang J. On the Unique Continuation Property for a Coupled Systems of Third-order Nonlinear Schrodinger Equations, Asian Journal of Mathematics and Computer Research. 2023;30(4):42-62. Article no. AJOMCOR.11706,

[8] Md. Asraful A, Md. Zaidur R. Stability Analysis of Modified General Version of Gauss-type Proximal Point Method for solving Generalized Equations Using Metrically Regular Mapping, Asian Journal of Mathematics and Computer Research. 2023;30(2):38-52. Article no. AJOMCOR.11536,

[9] Iricanin B, Stevic S. Some systems of non-linear difference equations of higher order with periodic solutions, Dynamic of Continuous, Discrete and Impulse Sysetems, Series A Mathematical Analysis. 2006;13:499-507.

[10] Cinar C, Yalcinkaya I. On the positive solutions of difference equation system, International Mathematical Journal. 2004;5.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://prh.ikprress.org/review-history/12235