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Analytic Representation of the Sequence of Functions on $L^1(\mathbb{R})$ Space

Egzona Iseni [a], Shpetim Rexhepi [a*], Bilall Shaini [b] and Ilir Demiri [a]

[a] Mother Teresa University, Skopje, North Macedonia.
[b] University of Tetovo, North Macedonia.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper studies the convergence of the functional sequence, of functions belonging to $L^1(\mathbb{R})$ space and their analytic representations. In the first theorem using Parseval's theorem and a result that describes the inverse Fourier transform between the Heaviside function, we prove the analytic representation of the functional sequence, of the functions belonging to the $L^1(\mathbb{R})$ space, and uniform convergence of the sequence of their analytic representation. Using Fourier transform and the Cauchy representation we show that the sequence of the analytic representation, converges uniformly to the functions belonging to the same space on the compact subset. In the last part, we applied Fubini's theorem to the functional sequence $\theta_n(t)$, and if we have a sequence of functions on $L^1(\mathbb{R})$ and another function from the same space, then the sequence of convolutions converges also on $L^1(\mathbb{R})$ space.

**Corresponding author: Email: shpetim.rexhepi@unt.edu.mk;*

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1 Introduction

In this paper, we will use the standard notations from the Schwartz distribution theory, taken from [1-10]. Denote by $C^{\infty}(\mathrm{R}^n)$ the space of all infinitely differentiable functions on $\mathbb{R}^n$. The subspace of $C^{\infty}(\mathbb{R}^n)$ that contains all the functions that have a compact support is denoted by $C_0^{\infty}(\mathbb{R}^n)$.

The space of functions belonging to $C_0^{\infty}(\mathbb{R}^n)$, that represents the set of all test functions is denoted by D. The convergence of the test functions is defined as follows:

A functional sequence $\{\eta_t\}$, $\eta_t \in D$ converges to the function η belonging to the space D as $t \to t_0$, if and only if there exists a compact set $A \subset \mathbb{R}^n$ such that $\operatorname{supp}(\eta_t) \subseteq A$, for every t, $\operatorname{supp}(t) \subseteq A$ and for every *n*-tiple *m*, where *m* is a positive integer or zero, the functional sequence $\{\theta^{(m)}\eta_t(x)\}$ converges uniformly to the function $\theta^{(m)}\eta(x)$ on the compact set A, as $t \to t_0$.

Definition 1. [1,3] Distribution T is a continuous linear functional on the space D. Instead of $T(\eta)$, in distribution theory, symbolically it is written as $< T, \eta >$, for the value of T acting on a test function η.

Denote by D' the space of all distributions.

Let η belong to one of the function spaces D or S, and θ be a function such that

$$< T_\theta, \eta >= \int_{\mathbb{R}^n} \theta(t)\eta(t)dt, \quad \eta \in S(\eta \in D),$$

exists and is less than infinity, then T_θ represents a regular distribution on the spaces *S* or *D*, generated by the function θ.

One of the first results is that if the function $\theta \in L^1(\mathbb{R})$, then the function

$$\hat{\theta}(z) = \frac{1}{2\pi i} < \theta(t), \frac{1}{t-z} >, \ z = x + iy, \ x \notin \operatorname{supp}\theta,$$

is Cauchy representation of the function θ, i.e.,

$$\lim_{y \to 0^+} \int_{-\infty}^{\infty} (\hat{\theta}(x+iy) - \hat{\theta}(x-iy))\eta(x)dx = \int_{-\infty}^{\infty} \theta(x)\eta(x)dx,$$

where $\eta \in D$, and D represents the Schwartz space of test functions.

To prove the theorems in the main results we will use some known properties of the Fourier transform of a function denoted by $F(\theta)$ and the inverse Fourier transform denoted by $F^{-1}(\theta)$. Also Parseval's formulas for the functions $\theta_1, \theta_2 \in L^2(\mathbb{R})$ or $\theta_1, \theta_2 \in L^1(\mathbb{R})$, are used. The following relations

$$\int_{-\infty}^{\infty} F(\theta_1, t)\theta_2(t)dt = \int_{-\infty}^{\infty} \theta_1(u)F(\theta_2, u)du,$$

and

$$\int_{-\infty}^{\infty} F^{-1}(\theta_1,t)\theta_2(t)dt = \int_{-\infty}^{\infty} \theta_1(u)F^{-1}(\theta_2,u)du\,,$$

are true.

In the papers [2] and [4] it is proved the following result: For the functions $g \in L^1(\mathbb{R})$ and $\theta(t) = F^{-1}(g,t)$ the Cauchy representation of the function θ is

$$\hat{\theta}(z) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{0} g(u)e^{-iwz}du, & y > 0 \\ -\frac{1}{2\pi}\int_{0}^{\infty} g(u)e^{-iwz}du, & y < 0 \end{cases}$$

To prove Theorem 1, also we are based on the following relations:

$$F^{-1}(H(t)e^{itz},u) = \frac{1}{2\pi(u-z)i}, \quad y > 0,$$
$$F^{-1}(H(-t)e^{itz},u) = \frac{-1}{2\pi(u-z)i}, \quad y < 0.$$

where $H(t)$ represents the Heaviside function.

The function $\psi(x) = (g*\theta)(x) = \int_{\mathbb{R}} \theta(x-y)g(y)dy$ is an element of the space $L^1(\mathbb{R})$, such that $\|\psi\|_1 \le \|\theta\|_1 \|g\|_1$ and $g*\theta = \theta*g$.

By ψ is denoted the convolution of the functions θ and g.

The following theorem is also well-known.

-Let the functions θ and g belong to the space L^1 and let $\psi = g*\theta = \theta*g$, then the function ψ has the Cauchy representation.

$$\hat{\psi}(z) = \frac{1}{2\pi i}\int \frac{\psi(u)}{u-z}du$$
$$= \int_{\mathbb{R}} \theta(u)\hat{g}(z-u)du =$$
$$= \int_{\mathbb{R}} g(u)\hat{\theta}(z-u)du, z = x+iy, \ \operatorname{Im} z \ne 0$$

2 Main Results

Theorem 1.

Let $\theta_n(t) = \frac{n\sin\left(\frac{t}{n}\right)}{t(1+t^2)}$ be a functional sequence that converges to the function $\theta(t)$.

Then:

[1] For every $n \in \mathbb{N}$, $\hat{\theta}_n(z) = \frac{1}{2\pi i} < \theta_n(t), \frac{1}{t-z} >, \ \mathrm{Im}\, z \neq 0$, is an analytic representation of f_n.

[2] The functional sequence $\left\{\hat{\theta}_n(z)\right\}$ uniformly converges on any compact subset of $\mathbb{C}/\mathbb{R}$ to the function $\hat{\theta}(z)$, where

$$\hat{\theta}(z) = \frac{1}{2\pi i} < \theta(t), \frac{1}{t-z} >, \ \mathrm{Im}\, z \neq 0 .$$

[3] The function $\hat{\theta}(z)$ is an analytic representation of the function $\theta(t)$,

where $\theta(t)$ is the limit of the functional sequence.

Proof

We will prove that $\lim_{n\to\infty} \theta_n(t) = \theta(t) = \frac{1}{1+t^2}$.

$$\lim_{n\to\infty} \theta_n(t) = \lim_{n\to\infty} \frac{n \sin\left(\frac{t}{n}\right)}{t\left(1+t^2\right)} = \lim_{n\to\infty} \frac{\sin\left(\frac{t}{n}\right)}{\frac{t}{n}(1+t^2)} = \frac{1}{1+t^2}$$

and also $\theta(t) = \frac{1}{1+t^2}$ is a continuous function, which implies measure function and element of the L^1 space.

i) Let $z = x + iy$ be a complex number such that $\mathrm{Im}\, z \neq 0$. For any $\eta \in D$ and $n \in \mathbb{N}$, we have:

$$\int_{-\infty}^{\infty} (\hat{\theta}_n(x+iy) - \hat{\theta}_n(x-iy))\eta(x)dx = = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{\frac{n\sin\left(\frac{t}{n}\right)}{t(1+t^2)}}{t-x-iy} - \frac{\frac{n\sin\left(\frac{t}{n}\right)}{t(1+t^2)}}{t-x+iy} \right) dt \right) \eta(x)dx.$$

Using the formulas in [2]

$$\frac{1}{2\pi i(t-x-iy)} = F^{-1}(H(w)e^{iw(x+iy)}, t), \ \ y > 0$$

And

$$\frac{-1}{2\pi i(t-x-iy)} = F^{-1}(H(-w)e^{iw(x+iy)}, t), \ \ y < 0$$

And using the Parseval's formula, we get the following results:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{n\sin(\frac{t}{n})}{t(1+t^2)}F^{-1}(H(w)e^{iw(x+iy)},t)dt+\int_{-\infty}^{\infty}\frac{n\sin(\frac{t}{n})}{t(1+t^2)}F^{-1}(H(-w)e^{iw(x-iy)},t)dt\right)\eta(x)dx=$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}H(w)e^{iw(x+iy)}F^{-1}(\frac{n\sin(\frac{t}{n})}{t(1+t^2)},w)dw+\int_{-\infty}^{\infty}H(-w)e^{iw(x-iy)}F^{-1}(\frac{n\sin(\frac{t}{n})}{t(1+t^2)},w)dw\right)\eta(x)dx=$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}H(w)e^{iwx}e^{-yw}F^{-1}(\frac{n\sin(\frac{t}{n})}{t(1+t^2)},w)dw+\int_{-\infty}^{\infty}H(-w)e^{iwx}e^{wy}F^{-1}(\frac{n\sin(\frac{t}{n})}{t(1+t^2)},w)dw\right)\eta(x)dx=$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{iwx}F^{-1}(\frac{n\sin(\frac{t}{n})}{t(1+t^2)},w)(H(w)e^{-yw}+H(-w)e^{wy})dw\right)\eta(x)dx.$$

Now using the Fubini's theorem, we obtain:

$$\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}e^{iwx}\eta(x)dx)F^{-1}(\frac{n\sin\left(\frac{t}{n}\right)}{t\left(1+t^2\right)},w)\left(H(w)e^{-yw}+H(-w)e^{wy}\right)dw$$

$$=\int_{-\infty}^{\infty}F(\eta,w)F^{-1}(\frac{n\sin\left(\frac{t}{n}\right)}{t\left(1+t^2\right)},w)\left(H(w)e^{-yw}+H(-w)e^{wy}\right)dw.$$

Next according to the Lebesgue theorem for the dominated convergence, we get the following:

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}F\left(\eta,w\right)F^{-1}\left(\frac{n\sin\left(\frac{t}{n}\right)}{t\left(1+t^2\right)},w)(H(w)e^{-yw}+H(-w)e^{wy}\right)dw$$

$$=\int_{-\infty}^{\infty}F\left(\eta,w\right)F^{-1}\left(\frac{n\sin\left(\frac{t}{n}\right)}{t\left(1+t^2\right)},w\right)dw.$$

Using again Parseval's formulas gives that: $\lim_{y\to 0^+}\int_{-\infty}^{\infty}\frac{n\sin(\frac{t}{n})}{t(1+t^2)}F^{-1}(F(\eta,t)dt=\int_{-\infty}^{\infty}\frac{n\sin(\frac{t}{n})}{t(1+t^2)}\eta(t)dt$

i.e., $\hat{\theta}_n(z)$ is an analytic representation of the sequence $\theta_n(t)$ for every $n\in\mathbb{N}$.

ii) Now let $z=x+iy$ be a complex number such that $\operatorname{Im} z\neq 0$. Taking difference between $\hat{\theta}_n(z)$ and $\hat{\theta}(z)$, we get that:

$$\hat{\theta}_n(z)-\hat{\theta}(z)=\frac{1}{2\pi i}\left(\int_{-\infty}^{\infty}\frac{\frac{n\sin(\frac{t}{n})}{t(1+t^2)}}{t-z}dt-\int_{-\infty}^{\infty}\frac{1/(1+t^2)}{t-z}dt\right).$$

Since $\theta_n(z),\hat{\theta}(z)\in L^1(\mathbb{R})$ for every $n\in\mathbb{N}$ and $\frac{1}{t-z}\in L^1(\mathbb{R})$ for $\operatorname{Im} z\neq 0$ and the properties for integral, we obtain:

$$\left|\hat{\theta}_n(z)-\hat{\theta}(z)\right|=\left|\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\frac{n\sin(\frac{t}{n})}{t(1+t^2)}}{t-z}dt-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1/(1+t^2)}{t-z}dt\right|\leq$$

$$\leq\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{n\sin(\frac{t}{n})}{t(1+t^2)}-1/(1+t^2)\right|}{|t-z|}dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{n\sin(\frac{t}{n})}{t(1+t^2)}-1/(1+t^2)\right|}{\sqrt{(t-x)^2+y^2}}dt\leq$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{n\sin(\frac{t}{n})}{t(1+t^2)}-1/(1+t^2)\right|}{y}dt\leq\frac{1}{2\pi\delta}\int_{-\infty}^{\infty}\left|\frac{n\sin(\frac{t}{n})}{t(1+t^2)}-\frac{1}{1+t^2}\right|dt.$$

By the assumption, the functional sequence $\theta_n(t)=\frac{n\sin\left(\frac{t}{n}\right)}{t\left(1+t^2\right)}$ converges to the function $\theta(t)=\frac{1}{1+t^2}$ in $L^1(\mathbb{R})$ as *n* tends to infinity. So, for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, we have:

$$\left|\int_{-\infty}^{\infty}\left(\frac{n\sin(\frac{t}{n})}{t(1+t^2)}-\frac{1}{1+t^2}\right)dt\right|<\varepsilon$$

On the other side $\frac{1}{|t-z|}$ is bounded, for $\operatorname{Im} z\neq 0$.

Finally, for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ and all $z\in\mathbb{C}$, for which $\operatorname{Im} z\neq 0$, we have

$$\left|\hat{\theta}_n(z)-\theta_n\right|<\varepsilon$$

The proof of iii) is similar to the proof of i).

Theorem 2. Let $\theta_n(x) = \dfrac{n\sin\left(\frac{x}{n}\right)}{x(1+x^2)}$ be a functional sequence and $g \in L^1$. Let $\psi_n = \theta_n * g$. Then the sequence $\{\psi_n\}$ converges to $\psi = \theta * g$ in L^1.

Proof. We know that $\theta_n(x) = \dfrac{n\sin\left(\frac{x}{n}\right)}{x(1+x^2)}$ and $\dfrac{1}{1+x^2}$ are elements of L^1. Let us consider the difference:

$$\left|\int_{\mathbb{R}} \psi_n(x)dx - \int_{\mathbb{R}} \psi(x)dx\right| =$$

$$= \left|\int_{\mathbb{R}} \left(\frac{n\sin\left(\frac{x}{n}\right)}{x(1+x^2)} * g\right)(x)dx - \int_{\mathbb{R}} \left(\frac{1}{1+x^2} * g\right)(x)dx\right|$$

$$= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{n\sin\left(\frac{y}{n}\right)}{y(1+y^2)} g(x-y)dy - \frac{1}{1+y^2} g(x-y)dy\right]dx\right|$$

$$= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{n\sin\left(\frac{y}{n}\right)}{y(1+y^2)} - \frac{1}{1+y^2}\right] g(x-y)dydx\right|.$$

By Fubini's Theorem, in the last integral, similar as in [5] we may change the order of integration and get that:

$$\left|\int_{\mathbb{R}} \psi_n(x)dx - \int_{\mathbb{R}} \psi(x)dx\right| \leq \int_{\mathbb{R}} \left|\frac{n\sin\left(\frac{y}{n}\right)}{y(1+y^2)} - \frac{1}{1+y^2}\right| dy \int_{\mathbb{R}} |g(x-y)| dx$$

Since $\theta_n \to \theta$ in L^1 and by Lebesgue's theorem for the dominated convergence we get that $\psi_n \to \psi$ in L^1.

3 Conclusions

Convergent sequences of functions in L^1 space and their analytic representations, prove that the sequence of the analytic representations uniformly converges on a compact subset of C/R, and gives additional results about the analytic representation of the boundary function. Also, the generalized Cauchy representation helps to obtain results concerning the analytic representation of the convolution of the functions.

In the future, we suggest if these results are valid in other functional spaces not only in $L^1(\mathbb{R})$ space, their analytic representations, and generalization of the results to analytic representations in distributional spaces.

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Competing Interests

Authors have declared that no competing interests exist.

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