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Oscillation Criteria for the Solutions of Second Order Half-linear Delay Dynamic Equations on Time Scales

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Authors' contributions

This work was carried out in collaboration between both authors. Author QZ designed the study, performed the statistical analysis, wrote the protocol, and managed literature searches. Author SL wrote the first draft of the manuscript, managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper, we are concerned with a class of second-order half-linear delay dynamic equations on time scales. By using the generalized Riccati technique and the integral averaging technique of Grace-type, four new oscillation criteria are obtained for every solutions of the equations to be oscillatory.

Keywords: Oscillation criterion; dynamic equations; time scale.

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1 Introduction

According to the important academic value and application background in Quantum Physics (especially in Nuclear Physics), engineering mechanics and control theory, oscillation theory of

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dynamic equations on time scales has become one of the research hotspots now. The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1], in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory, see [2-4]. A time scale $\mathbf{T}$ is an arbitrary closed subset of the reals, and when this time scale is equal to the reals or to the integers the cases represent the classical theories of differential and difference equations. The oscillation theory of dynamic equations on time scales not only unifies oscillation theories of differential equations and difference equations but also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbf{T} = q^{\mathbf{N}_0} = \{q^t : t \in \mathbf{N}_0,\ q > 1\}$ or $\mathbf{T} = \overline{q^{\mathbf{Z}}} = q^{\mathbf{Z}} \cup \{0\}$ (which has important applications in quantum theory). For another example, when $\mathbf{T} = h\mathbf{N}, \mathbf{T} = \mathbf{N}^2 = \{t^2 : t \in \mathbf{N}\}$ and $\mathbf{T} = \mathbf{T}_n = \{t_n = \sum_{k=1}^{n} \frac{1}{k},\ n \in \mathbf{N}_0\}$, it can be applied to dynamic equations different from the differential equations and difference equations. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to Bohner [5], Erbe [6,7], and [8-16]. In this paper we deal with the oscillatory behavior of all solutions of half-linear second-order delay dynamic equation

$$(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0, \tag{1.1}$$

subject to the hypotheses:

(H_1) $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above, and $t_0 \in \mathbb{T}$ with $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

(H_2) $\gamma \geq 1$ is the ratio of two positive odd integers.

(H_3) $a,\ q$ are positive real-valued right-dense continuous functions on an arbitrary time scale $\mathbb{T}$.

(H_4) $\tau \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ is a strictly increasing and differentiable function such that $\tau(t) \leq t$, $\tau(t) \to \infty$ as $t \to \infty$ and $\tau(\mathbb{T}) = \mathbb{T}$.

(H_5) $f \in C(\mathbb{R}, \mathbb{R})$ is a continuous function such that satisfies for some positive constant L,

$$\frac{f(x)}{x^{\gamma}} \geq L \ \texttt{for all} \ x \neq 0.$$

By a solution of (1.1), we mean a nontrivial real-valued function x satisfying (1.1) for $t \in \mathbb{T}$. We recall that a solution x of equation (1.1) is said to be oscillatory on $[t_0, \infty)_{\mathbb{T}}$ in case it is neither eventually positive nor eventually negative; otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. Our attention is restricted on those solutions x of (1.1) which x is not the eventually identically zero. Since $a(t) > 0$, we shall consider both the cases

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \triangle t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \triangle t < \infty. \tag{1.3}$$

It is easy to see that (1.1) can be transformed into a second-order nonlinear delay dynamic equation

$$(a(t)x^{\triangle}(t))^{\triangle} + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.4}$$

where $\gamma = 1$. In (1.1), if $f(x) = x^{\gamma}$, $\tau(t) = t$, then (1.1) is simplified to an equation

$$(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} + q(t)x^{\gamma}(t) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.5}$$

In (1.4), if $a(t) = 1$, then (1.4) is simplified to an equation

$$x^{\triangle\triangle}(t) + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.6}$$

In (1.6), if $f(x) = x$, then (1.6) is simplified to an equation

$$x^{\triangle\triangle}(t) + q(t)x(\tau(t)) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.7}$$

In 2005, Agarwal, Bohner and Saker [12] considered the linear delay dynamic equations (1.7), Sahiner[13] considered the nonlinear delay dynamic equations(1.6),and established some sufficient conditions for oscillation of (1.7) and (1.6). In 2007, Erbe, Peterson and Saker [14] considered the general nonlinear delay dynamic equations (1.4), setting to obtain some new oscillation criteria which improve the results given by Sahiner [13]. In 2005, Saker [15] and in 2009, Grace, Bohner and Agarwal [10] considered the half-linear dynamic equations (1.5), and established some sufficient conditions for oscillation of (1.5). In 2009, Sun, Han and Zhang [8] extended and improved the results of [12-14] to (1.1), meanwhile obtained some oscillatory criteria of (1.1). On this basis, we continue to discuss the oscillation of solutions of the equation (1.1), by using the generalized Riccati transformation and the inequality technique, we obtain some new oscillation criteria of the equation (1.1), our results extend and improve some known results.

The paper is organized as follows: In Sect. 2 we give several useful formulas. In Sect. 3, we give several lemmas. In Sect. 4, we intend to use the generalized Riccati transformation and the inequality technique, to obtain some sufficient conditions for oscillation of all solutions of equation (1.1).

2 Some Preliminaries

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g

$$(fg)^{\triangle}(t) = f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) = f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)), \tag{2.1}$$

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}. \tag{2.2}$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of $f^{\triangle}$ is defined by

$$\int_b^c f^{\triangle}(t)\triangle t = f(c) - f(b).$$

The integration by parts formula reads

$$\int_b^c f^{\triangle}(t)g(t)\triangle t = f(c)g(c) - f(b)g(b) - \int_b^c f^{\sigma}(t)g^{\triangle}(t)\triangle t \tag{2.3}$$

and infinite integrals are defined by

$$\int_b^{\infty} f(s)\triangle s = \lim_{t\to\infty}\int_b^t f(s)\triangle s.$$

For more details, see [3,4].

3 Several Lemmas

Lemma 3.1. *(Bohner et al. [3, Theorem 1.93]) Assume that $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^{\triangle}(t)$ and $w^{\widetilde{\triangle}}(v(t))$ exist for $t \in \mathbb{T}^k$, where*

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \boldsymbol{if} \ \ \sup\mathbb{T} < \infty, \\ \mathbb{T}, & \boldsymbol{if} \ \ \sup\mathbb{T} = \infty. \end{cases}$$

Then

$$(w \circ v)^{\triangle} = (w^{\widetilde{\triangle}} \circ v)v^{\triangle}. \tag{3.1}$$

Lemma 3.2. *(Bohner et al. [3, Theorem 1.90]) Assume that x is delta-differentiable and eventually positive or eventually negative, then*

$$((x(t))^{\gamma})^{\triangle} = \gamma \int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{\gamma-1} x^{\triangle}(t)\mathrm{d}h. \tag{3.2}$$

Lemma 3.3. *(Sun et al. [8, Lemma 2.1]) Assume (H_1)-(H_5) and (1.2). Let x be an eventually position solution of (1.1). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$x^{\triangle}(t) > 0, \quad (a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.3}$$

4 Main Results

Theorem 4.1. *Assume that the conditions (H_1)-(H_5) and (1.2) hold, if there exists a positive nondecreasing $\triangle-$differentiable function $\delta \in \mathrm{C}^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that for every $T \in [t_0, \infty)_{\mathbb{T}}$*

$$\limsup_{t\to\infty} \int_T^t \Big[L\delta(s)q(s) - \eta^{\gamma}(\tau(s))\delta^{\triangle}(s)\Big] \triangle s = \infty, \tag{4.1}$$

where

$$\eta(\tau(t)) = \left(\int_T^t \left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}} \tau^{\triangle}(s)\triangle s\right)^{-1}. \tag{4.2}$$

Then (1.1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that x is a nonoscillatory solutions of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}},\ t_1 \in [t_0, \infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when x is eventually negative is similar. By Lemma 3.3 we have $x^{\triangle}(t) > 0$ for all $t \in [t_2, \infty)_{\mathbb{T}},\ t_2 \in [t_1, \infty)_{\mathbb{T}}$, and by Lemma 3.1 and ($H_4$), there exists $T \in [t_2, \infty)_{\mathbb{T}}$ such that $(x(\tau(t)))^{\triangle} > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. Using (3.2) and (3.3), we have

$$\begin{aligned} ((x(\tau(t)))^{\gamma})^{\triangle} &= \gamma \int_0^1 [h(x(\tau(t)))^{\sigma} + (1-h)x(\tau(t))]^{\gamma-1} (x(\tau(t)))^{\triangle}\mathrm{d}h \\ &\geq \gamma \int_0^1 [h(x(\tau(t))) + (1-h)x(\tau(t))]^{\gamma-1} (x(\tau(t)))^{\triangle}\mathrm{d}h \\ &= \gamma(x(\tau(t)))^{\gamma-1}(x(\tau(t)))^{\triangle}. \end{aligned}$$

Let $v = \tau, w = x$ in Lemma 3.1, we know that $\mathbb{T}$ is unbounded above by (H_1), which implies $\mathbb{T}^k = \mathbb{T}$. Further, as $\widetilde{\mathbb{T}} = v(\mathbb{T}) = \tau(\mathbb{T}) = \mathbb{T}$ by (H_4), we can get from Lemma 3.1 that

$$(x(\tau(t)))^{\triangle} = x^{\triangle}(\tau(t))\tau^{\triangle}(t).$$

Thus

$$((x(\tau(t)))^{\gamma})^{\triangle} \geq \gamma(x(\tau(t)))^{\gamma-1}x^{\triangle}(\tau(t))\tau^{\triangle}(t) > 0. \tag{4.3}$$

Define the function $W(t)$ by

$$W(t)=\delta(t)\frac{a(t)(x^{\triangle}(t))^{\gamma}}{(x(\tau(t)))^{\gamma}},\quad t\in[T,\infty)_{\mathbb{T}}. \tag{4.4}$$

Then on $[T,\infty)_{\mathbb{T}}$, we have $W(t)>0$, by (2.1), (2.2) and (1.1), we obtain

$$\begin{aligned}
W^{\triangle}(t)&=\frac{\delta(t)}{(x(\tau(t)))^{\gamma}}\left(a(t)(x^{\triangle}(t))^{\gamma}\right)^{\triangle}+a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\frac{(x(\tau(t)))^{\gamma}\delta^{\triangle}(t)-\delta(t)\left((x(\tau(t)))^{\gamma}\right)^{\triangle}}{(x(\tau(t)))^{\gamma}(x(\tau(\sigma(t))))^{\gamma}}\\
&\le -Lq(t)\delta(t)+\frac{a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\delta^{\triangle}(t)}{(x(\tau(\sigma(t))))^{\gamma}}-\frac{\delta(t)a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\left((x(\tau(t)))^{\gamma}\right)^{\triangle}}{(x(\tau(t)))^{\gamma}(x(\tau(\sigma(t))))^{\gamma}} \qquad (4.5)\\
&\le -Lq(t)\delta(t)+\frac{a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\delta^{\triangle}(t)}{(x(\tau(\sigma(t))))^{\gamma}}\\
&\le -Lq(t)\delta(t)+\frac{a(\tau(t))(x^{\triangle}(\tau(t)))^{\gamma}\delta^{\triangle}(t)}{(x(\tau(t)))^{\gamma}}=-Lq(t)\delta(t)+a(\tau(t))\delta^{\triangle}(t)\left(\frac{x^{\triangle}(\tau(t))}{x(\tau(t))}\right)^{\gamma}. \qquad (4.6)
\end{aligned}$$

Now

$$\begin{aligned}
x(\tau(t))&=x(\tau(T))+\int_T^t x^{\triangle}(\tau(s))\tau^{\triangle}(s)\triangle s\\
&=x(\tau(T))+\int_T^t\left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}}(a(\tau(s)))^{\frac{1}{\gamma}}x^{\triangle}(\tau(s))\tau^{\triangle}(s)\triangle s\\
&\ge\left(\int_T^t\left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}}\tau^{\triangle}(s)\triangle s\right)\left(a(\tau(t))\left(x^{\triangle}(\tau(t))\right)^{\gamma}\right)^{\frac{1}{\gamma}},
\end{aligned}$$

and thus

$$\left(\frac{x^{\triangle}(\tau(t))}{x(\tau(t))}\right)^{\gamma}\le\frac{1}{a(\tau(t))}\left(\int_T^t\left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}}\tau^{\triangle}(s)\triangle s\right)^{-\gamma}=\frac{\eta^{\gamma}(\tau(t))}{a(\tau(t))}\quad\texttt{for}\quad t\in[T,\infty)_{\mathbb{T}}. \tag{4.7}$$

Using (4.7) in (4.6), we have

$$W^{\triangle}(t)\le -Lq(t)\delta(t)+\eta^{\gamma}(\tau(t))\delta^{\triangle}(t)\quad\texttt{on}\quad[T,\infty)_{\mathbb{T}}. \tag{4.8}$$

Integrating (4.8) from T to t, we obtain

$$0<W(t)\le W(T)-\int_T^t\left[L\delta(s)q(s)-\eta^{\gamma}(\tau(s))\delta^{\triangle}(s)\right]\triangle s,$$

which gives a contradiction using (4.1). This completes the proof. □

Now, when (1.3) holds, we give the condition that guarantee that every solution of (1.1) oscillates.

Theorem 4.2. *Assume that the conditions (H_1)-(H_5) and (1.3) hold. Let δ is as in Theorem 4.1 and conditions (4.1),(4.2) hold, if for every $T\in[t_0,\infty)_{\mathbb{T}}$*

$$\int_T^{\infty}\left[\frac{1}{a(s)}\int_T^s\theta^{\gamma}(u)q(u)\triangle u\right]^{\frac{1}{\gamma}}\triangle s=\infty, \tag{4.9}$$

where

$$\theta(t)=\int_t^{\infty}\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}\triangle s. \tag{4.10}$$

Then (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that x is a nonoscillatory solutions of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t)>0$ and $x(\tau(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}},\ t_1\in[t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when x is eventually negative is similar. Since $a(t)(x^{\triangle}(t))^{\gamma}$ is decreasing for all $t\in[T,\infty)_{\mathbb{T}},\ T\in[t_1,\infty)_{\mathbb{T}}$, it is eventually of one sign and hence $x^{\triangle}(t)$ is eventually of one sign. Thus, we shall distinguish the following two cases:

(I) $x^{\triangle}(t)>0$ for $t\geq T$; and
(II) $x^{\triangle}(t)<0$ for $t\geq T$.

Case (I). The proof when $x^{\triangle}(t)$ is an eventually positive is similar to that of the proof of Theorem 4.1 and it hence is omitted.

Case (II). For $s\geq t\geq T$, we have

$$a(s)(-x^{\triangle}(s))^{\gamma}\geq a(t)(-x^{\triangle}(t))^{\gamma},$$

and hence

$$-x^{\triangle}(s)\geq\left(\frac{a(t)}{a(s)}\right)^{\frac{1}{\gamma}}(-x^{\triangle}(t)). \tag{4.11}$$

Integrating (4.11) from $t\geq T$ to $u\geq t$ and letting $u\to\infty$ yields

$$x(t)\geq\left[\int_t^{\infty}\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}\triangle s\right](a(t))^{\frac{1}{\gamma}}(-x^{\triangle}(t))=-\theta(t)a^{\frac{1}{\gamma}}(t)x^{\triangle}(t)\ \ \texttt{for}\ \ t\in[T,\infty)_{\mathbb{T}},$$

and thus

$$(x(t))^{\gamma}\geq-(\theta(t))^{\gamma}a(t)(x^{\triangle}(t))^{\gamma}\geq-(\theta(t))^{\gamma}a(T)(x^{\triangle}(T))^{\gamma}=b(\theta(t))^{\gamma}\ \ \texttt{for}\ \ t\in[T,\infty)_{\mathbb{T}}, \tag{4.12}$$

with $b=-a(T)(x^{\triangle}(T))^{\gamma}>0$. Using (4.12) in equation (1.1), we find

$$-(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle}\geq Lq(t)(x(\tau(t)))^{\gamma}\geq Lq(t)(x(t))^{\gamma}\geq bL\theta^{\gamma}(t)q(t)\ \ \texttt{for}\ \ t\in[T,\infty)_{\mathbb{T}}, \tag{4.13}$$

Integrating (4.13) from T to t, we have

$$-a(t)(x^{\triangle}(t))^{\gamma}\geq-a(T)(x^{\triangle}(T))^{\gamma}+bL\int_T^t\theta^{\gamma}(s)q(s)\triangle s\geq bL\int_T^t\theta^{\gamma}(s)q(s)\triangle s,$$

so that

$$-x^{\triangle}(t)\geq\left[\frac{bL}{a(t)}\int_T^t\theta^{\gamma}(s)q(s)\triangle s\right]^{\frac{1}{\gamma}}. \tag{4.14}$$

Integrating (4.14) from T to t, we obtain

$$\infty>x(T)\geq-x(t)+x(T)\geq\int_T^t\left[\frac{bL}{a(s)}\int_T^s\theta^{\gamma}(u)q(u)\triangle u\right]^{\frac{1}{\gamma}}\triangle s\to\infty\ \ \texttt{as}\ \ t\to\infty$$

by (4.9), a contradiction. This completes the proof. □

Remark 4.1. Suppose that the condition (1.3) is satisfied, then Theorem 4.2 obtains the sufficient condition of oscillation for equation (1.1). The usual result is that the conditions (1.3) was established, then every solution of the equation (1.1) is either oscillatory or converges to zero on $[t_0,\infty)_{\mathbb{T}}$.

Theorem 4.3. *Assume that the conditions (H_1)-(H_5) and (1.2) hold, if there exists a positive $\triangle-$differentiable function $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that for every $T \in [t_0,\infty)_{\mathbb{T}}$*

$$\limsup_{t\to\infty}\int_T^t \left[Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}}(\delta^{\triangle}(s))^2}{4\gamma\delta(s)\tau^{\triangle}(s)}(\eta^{\sigma}(s))^{\gamma-1}\right]\triangle s = \infty, \tag{4.15}$$

where η is as in (4.2). Then (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that x is a nonoscillatory solutions of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}},\ t_1 \in [t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when x is eventually negative is similar. Proceeding as in the proof of Theorem 4.1, we obtain (4.3) and (4.5). Using (4.3) in (4.5), we have on $[T,\infty)_{\mathbb{T}}$ that

$$W^{\triangle}(t) \le -Lq(t)\delta(t) + \frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}x^{\triangle}(\tau(t))\tau^{\triangle}(t)}{(x(\tau(\sigma(t))))^{\gamma+1}}. \tag{4.16}$$

By $(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} < 0$, we have

$$a(\tau(t))(x^{\triangle}(\tau(t)))^{\gamma} \ge a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma} \quad \texttt{i.e.} \quad x^{\triangle}(\tau(t)) \ge \frac{(a(\sigma(t)))^{\frac{1}{\gamma}}}{(a(\tau(t)))^{\frac{1}{\gamma}}}x^{\triangle}(\sigma(t)). \tag{4.17}$$

Substituting (4.17) in (4.16), we obtain

$$W^{\triangle}(t) \le -Lq(t)\delta(t) + \frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^{\frac{\gamma+1}{\gamma}}}(W(\sigma(t)))^{\frac{\gamma+1}{\gamma}} \quad \texttt{on} \quad [T,\infty)_{\mathbb{T}}.$$

i.e., on $[T,\infty)_{\mathbb{T}}$

$$W^{\triangle}(t) \le -Lq(t)\delta(t) + \frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^{\frac{\gamma+1}{\gamma}}}(W(\sigma(t)))^{\frac{1-\gamma}{\gamma}}(W(\sigma(t)))^2. \tag{4.18}$$

Now inequality (4.7), i.e.

$$\frac{x(\tau(t))}{x^{\triangle}(\tau(t))} \ge \frac{a^{\frac{1}{\gamma}}(\tau(t))}{\eta(\tau(t))} \quad \texttt{i.e.} \quad \frac{x(t)}{x^{\triangle}(t)} \ge \frac{a^{\frac{1}{\gamma}}(t)}{\eta(t)}$$

implies on $[T,\infty)_{\mathbb{T}}$ that

$$(W(t))^{\frac{1-\gamma}{\gamma}} = (\delta(t)a(t))^{\frac{1-\gamma}{\gamma}}\left(\frac{x(t)}{x^{\triangle}(t)}\right)^{\gamma-1} \ge (\delta(t)a(t))^{\frac{1-\gamma}{\gamma}}\frac{a^{\frac{\gamma-1}{\gamma}}(t)}{\eta^{\gamma-1}(t)} = \delta^{\frac{1-\gamma}{\gamma}}(t)\eta^{1-\gamma}(t). \tag{4.19}$$

Using (4.19) in (4.18), we have on $[T,\infty)_{\mathbb{T}}$ that

$$\begin{aligned}
W^{\triangle}(t) &\le -Lq(t)\delta(t) + \frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^2}(\eta(\sigma(t)))^{1-\gamma}(W(\sigma(t)))^2 \\
&= -Lq(t)\delta(t) + \frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^{\triangle}(t))^2}{4\gamma\delta(t)\tau^{\triangle}(t)}(\eta(\sigma(t)))^{\gamma-1} \\
&\quad - \left[\frac{\sqrt{\gamma\delta(t)\tau^{\triangle}(t)(\eta(\sigma(t)))^{1-\gamma}}}{\delta(\sigma(t))\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}}W(\sigma(t)) - \frac{\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}\delta^{\triangle}(t)}{2\sqrt{\gamma\delta(t)\tau^{\triangle}(t)(\eta(\sigma(t)))^{1-\gamma}}}\right]^2 \\
&\le -\left[Lq(t)\delta(t) - \frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^{\triangle}(t))^2}{4\gamma\delta(t)\tau^{\triangle}(t)}(\eta(\sigma(t)))^{\gamma-1}\right].
\end{aligned}$$

Integrating both sides of this inequality from T to t, taking the lim sup of the resulting inequality as $t \to \infty$ and applying condition (4.15), we obtain a contradiction to the fact that $W(t) > 0$ for $t \in [T, \infty)_{\mathbb{T}}$. This completes the proof. □

Using the same ideas as in the proof of Theorem 4.2, when (1.3) holds, we can now obtain following result.

Theorem 4.4. *Assume that the conditions (H_1)-(H_5), (1.3), (4.9) and (4.15) hold, let η and θ be as in (4.2) and (4.10). Then (1.1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.*

Remark 4.2. Our results in this paper improve and extend some theorems in [15] and [10]. Especially, when $f(x) = x^\gamma$, $\tau(t) = t$, the Theorems 4.1-4.4 can convert into Theorems 3.2, 3.1, 3.6 and 3.5 in [10].

Here, we shall reformulate the above conditions which are sufficient for the oscillation of (1.1) when (1.2) holds on different time scales:

If $\mathbb{T} = \mathbb{R}$, then (1.1) becomes

$$(a(t)(x'(t))^\gamma)' + q(t)f(x(\tau(t))) = 0, \quad t \in [t_0, \infty), \tag{4.20}$$

condition (4.1) and (4.15), respectively, becomes

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[L\delta(s)q(s) - \delta'(s) \left(\int_{\tau(t_1)}^{\tau(s)} \left(\frac{1}{a(u)} \right)^{\frac{1}{\gamma}} \mathrm{d}u \right)^{-\gamma} \right] \mathrm{d}s = \infty, \tag{4.21}$$

and

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}} (\delta'(s))^2}{4\gamma\delta(s)\tau'(s)} \left(\int_{\tau(t_1)}^{\tau(s)} \left(\frac{1}{a(u)} \right)^{\frac{1}{\gamma}} \mathrm{d}u \right)^{1-\gamma} \right] \mathrm{d}s = \infty, \tag{4.22}$$

conditions (4.21) and (4.22) are new. The Theorems 4.1 and 4.3 are new even for the cases $\mathbb{T} = \mathbb{R}$.

If $\mathbb{T} = \mathbb{Z}$, then (1.1) becomes

$$\triangle(a_n(\triangle x_n)^\gamma) + q_n f(x_{n-\sigma}) = 0, \quad n = 0, 1, 2, \cdots, \tag{4.23}$$

condition (4.1) and (4.15), respectively, becomes

$$\limsup_{n\to\infty} \sum_{l=n_0}^{n} \left[Lq_l\delta_l - \triangle\delta_l \left(\sum_{k=n_0-\sigma}^{l-\sigma-1} \left(\frac{1}{a_k} \right)^{\frac{1}{\gamma}} \right)^{-\gamma} \right] = \infty, \tag{4.24}$$

and

$$\limsup_{n\to\infty} \sum_{l=n_0}^{n} \left[Lq_l\delta_l - \frac{(a_{l-\sigma})^{\frac{1}{\gamma}} (\triangle\delta_l)^2}{4\gamma\delta_l} \left(\sum_{k=n_0-\sigma}^{l-\sigma} \left(\frac{1}{a_k} \right)^{\frac{1}{\gamma}} \right)^{1-\gamma} \right] = \infty, \tag{4.25}$$

conditions (4.24) and (4.25) are new. The Theorem 4.1 and 4.3 are new even for the cases $\mathbb{T} = \mathbb{Z}$.

Example 4.5. *Considered the second-order half-linear delay 2-difference equations*

$$\left(t^{\frac{2}{3}} \left(x^{\triangle}(t) \right)^{\frac{5}{3}} \right)^{\triangle} + t^{-\frac{8}{5}} \left(x\left(\frac{t}{2} \right) \right)^{\frac{5}{3}} \left(1 + x^2\left(\frac{t}{2} \right) \right) = 0, \quad t \in \overline{2^{\mathbb{Z}}}, \quad t \ge t_0 := 2. \tag{4.26}$$

Here

$$a(t) = t^{\frac{2}{3}}, \quad q(t) = t^{-\frac{8}{5}}, \quad f(x) = x^{\frac{5}{3}} \left(1 + x^2 \right), \quad \tau(t) = \frac{t}{2}, \quad \gamma = \frac{5}{3}.$$

Then $\mathbb{T} = \overline{2^{\mathbb{Z}}}$ *is unbounded above, the conditions* (H_1)-(H_4) *are clearly satisfied,* (H_5) *holds with* $L = 1$, *next, for* $t \geq 2$ *so that*

$$\int_2^t \left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}} \triangle s = \int_2^t s^{-\frac{2}{5}} \triangle s = \frac{t^{\frac{3}{5}} - 2^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \to \infty \quad as \quad t \to \infty,$$

hence (1.2) is satisfied. Now let $\delta(t) = t^{\frac{3}{5}}$ *for all* $t > s \geq 2$,

$$\eta(\tau(t)) = \left(\int_2^t \left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}} \tau^{\triangle}(s)\triangle s\right)^{-1} = \left(\frac{1}{2^{\frac{3}{5}}}\int_2^t s^{-\frac{2}{5}}\triangle s\right)^{-1} = 2^{\frac{3}{5}}\frac{2^{\frac{3}{5}} - 1}{t^{\frac{3}{5}} - 2^{\frac{3}{5}}},$$

then

$$\int_2^t \left[L\delta(s)q(s) - \eta^{\gamma}(\tau(s))\delta^{\triangle}(s)\right]\triangle s = \int_2^t \left[\frac{1}{s} - 2\left(\frac{2^{\frac{3}{5}} - 1}{s^{\frac{3}{5}} - 2^{\frac{3}{5}}}\right)^{\frac{5}{3}}\left(s^{\frac{3}{5}}\right)^{\triangle}\right]\triangle s \to \infty \quad as \quad t \to \infty,$$

so that (4.1) is satisfied as well. Altogether, by Theorem 4.1, the equation (4.26) is oscillatory.

5 Conclusion

This paper is concerned with the oscillation of a class of second-order half-linear delay dynamic equations on time scales. By using the generalized Riccati technique and the integral averaging technique, four new oscillation criteria are obtained for every solutions of the equation (1.1) to be oscillatory. At the end of this paper an examples is given to illustrate our main results. Our results are new even both in the continuous case and discrete case.

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Competing Interests

The authors declare that no competing interests exist.

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